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Unitary representations of the Poincaré group

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Chapter I. Representations

1. Generalities about Banach- and Hilbert spaces

Usually, we consider only vector spaces over the field of real or complex numbers. If E, F are two vector spaces, we denote by $\mathrm{Hom}(E, F)$ the space of all (real- or complex-) linear maps. In the case $E = F$ we write $\mathrm{Hom}(E, E) = \mathrm{End}(E)$. The group of all invertible operators in $\mathrm{End}(E, E)$ is denoted by $\mathrm{GL}(E)$.

A norm on a vector space E is a real valued function $\|\cdot\|$ on E with the properties $\|a\| \ge 0$ and $= 0$ only for $a = 0$, $\|Ca\| = |C|\|a\|$, $\|a+b\| \le \|a\| + \|b\|$ $(a, b \in E)$, $C \in \mathbb{C}(or\mathrm{R}\ ))$. Then $\|a - b\|$ is a metric on E. The normed space E is called complete, or a Banach space, if every Cauchy sequence converges. Every normed space E can be embedded into a Banach space $\bar{E}$ as a dense subspace (with the restricted norm) in an essentially unique manner. One calls $\bar{E}$ the completion of E. Let $F \subset E$ be a linear subspace of a Banach space. It is a closed subspace if and only if it is a Banach space (with respect to the restricted norm). The closure of a linear subspace in a Banach space is a linear subspace and hence a Banach space. It can be identified with its completion. Since any two norms on a finite dimensional vector space are equivalent, every finite dimensional normed vector space is a Banach space. As a consequence, every finite dimensional subspace of a normed vector space is closed.

A linear map $A : E \to F$ between normed vector spaces is called *bounded* if there exists a constant $C \ge 0$ such that $\|Aa\| \le C\|a\|$ for all $a \in E$. Then there exists a smallest number C with this property. It is called the norm of A and is denoted by $\|A\|$. We mention that A is bounded if and only if it is continuous (at the origin is enough). For finite dimensional E, F each linear map is bounded. Let E be a normed space and F be a Banach space. The subspace of all bounded operators

$$B(E, F) \subset \mathrm{Hom}(E, F)$$

of $\mathrm{Hom}(E, F)$ is a Banach space (equipped with the operator norm). We use the abbreviation

$$B(E) = B(E, E).$$

If F is the ground field ($\mathbb{R}$ or $\mathbb{C}$) then $E' = B(E, F)$ is the so called dual space.

All what we have said so far about Banach spaces can be formulated and is true for real and complex Banach spaces. Now we consider complex vector spaces.

A hermitian form on a complex vector space E is a function $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ which is linear in the first variable and which has the property $\langle a, b \rangle = \overline{\langle b, a \rangle}$. It is called positive definite if $\langle a, a \rangle > 0$ for all $a \neq 0$. Then $\|a\| := \sqrt{\langle a, a \rangle}$ is norm. We call $(E, \langle \cdot, \cdot \rangle)$ a Hilbert space if it is a Banach space with this norm.

We will make use of the Theorem of Riesz:

Let $L : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space H. Then there exists a unique vector $a \in H$ such that $L(x) = \langle x, a \rangle$ (and each linear functional of this kind is continuous and has the norm $\|L\| = \|a\|$).

These special linear forms show that for every vector $a \in H$, $a \neq 0$, there exists a continuous linear functional L with the property $L(a) \neq 0$.

This statement is also true for Banach spaces. From the theorem of Hahn-Banach follows the following result:

For each non-zero vector $a \in E$ of a Banach space there exists a continuous linear functional L with the property $L(a) \neq 0$.

We will make use of another important result about Hilbert spaces. Let $A \subset H$ be a closed linear subspace. Denote by

$$B = \{b \in H;\ \langle a, b \rangle = 0 \text{ for all } a \in A\}$$

the orthogonal complement of A. This is a closed linear subspace and one has $H = A \oplus B$.

A family $(a_i)_{i \in I}$ is called an orthonormal system if any two members with different indices are orthogonal and if the norm of each member is one. A *Hilbert space basis* is by definition a maximal orthonormal system. It is easy to show (using Zorn's lemma and the above remark about orthogonal complements) that Hilbert space bases exist. Even more, every orthonormal system is contained in a maximal one.

A Hilbert space H is called separable if it contains a countable dense subset. One can show that this is the case if and only if each Hilbert space basis is finite or countable.

We recall some basics about infinite series. A series $a_1 + a_2 + \cdots$ in a Banach space E is called convergent if there exist a such that

$$\|a - \sum_{\nu=1}^{n} a_\nu\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.$$

A sufficient condition is that $\sum \|a_i\|$ converges. But this condition is not necessary.

In the special case that $E = H$ is a Hilbert space and that the a_i are pairwise orthogonal one can show the following. The series converges if and only if $\sum \|a_\nu\|^2$ converges.

We give an example of a separable Hilbert space. The space ℓ^2 consists of all sequences $(a_1, a_2, \ldots)$ of complex numbers such that $\sum |a_n|^2$ converges. It can be shown that for two $a, b \in \ell^2$ the series

$$\langle a, b\rangle = \sum a_n \bar{b}_n$$

converges absolutely and equips ℓ^2 with the structure as a Hilbert space. The usual unit vectors (1 at one place and 0 at the others) give a Hilbert space basis.

Let now H be any infinite dimensional separable Hilbert space with a Hilbert space basis $e_1, e_2, \ldots$. For each $a \in \ell^2$ the series

$$\sum_{n=1}^{\infty} a_n e_n := \lim_{N\to\infty} \sum_{n=1}^{N} a_n e_n$$

then converges in H. This gives a map

$$\ell^2 \stackrel{\sim}{\longrightarrow} H.$$

This map is actually an isomorphism of Hilbert spaces (which means that it is an isomorphism of vector spaces which preserves the Hermitian forms). Hence all infinite dimensional separable Hilbert spaces are isomorphic as Hilbert spaces. (The same kind of argument shows a standard result of linear algebra, namely that two finite dimensional Hilbert spaces are isomorphic as Hilbert spaces if and only if their dimensions agree.)

Assume that H_1, $H_2, \ldots$ is a sequence of pairwise orthogonal closed subspaces of the Hilbert space H. Assume that their algebraic sum is dense in H. If we choose a Hilbert space basis in each H_i and collect them, we get a Hilbert space basis of H. This shows that every $a \in H$ has a unique representation as convergent series $a = a_1 + a_2 + \cdots$ where $a_i \in H_i$. Recall that this means that $\sum_i \|a_i\|^2$ converges. We write this as

$$H = \widehat{\bigoplus_i} H_i$$

and call this a direct Hilbert sum.

There is an abstract version of this. Let H_n be a family of Hilbert spaces. We define H to be the set of all sequences (h_n), $h_n \in H_n$ such that $\sum \|h_n\|^2$ converges. There is a natural imbedding of H_n into H. The image $\tilde{H}_n$ consists of all elements of H such only the nth component can be different from 0. The space H carries a natural structure as Hilbert space and it is the direct Hilbert of the $\tilde{H}_n$. Usually one identifies $\tilde{H}_n$ with H_n and calls H the direct Hilbert sum of the H_n.

2. Generalities about measure theory

All topological spaces that carry measures are assumed to be Hausdorff and to have a countable basis of the topology. The latter means that there exists a countable system of open subsets such that each open subset can be written as a union of sets from this system. Every metric space with an countable dense subset (for example $\mathbb{C}^n$) has this property. Every subspace (equipped with the induced topology) keeps this property.

We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on a locally compact space X and by $\mathcal{C}_c(X)$ the subset of all continuous functions with compact support. A Radon measure is a linear functional $I : \mathcal{C}_c(X) \to \mathbb{R}$ which is real in the sense $I(\bar{f}) = \overline{I(f)}$ and positive in the sense that $I(f) \geq 0$ for real $f \geq 0$. Usually one writes

$$I(f) = \int_X f(x)dx.$$

We assume that the reader is familiar with some way to extend a Radon measure to the class of integrable functions. We just indicate the steps, how this can be done.

One introduces $\mathbb{R} \cup \{\infty\}$ as ordered set ($x \leq \infty$ for all x). Every non-empty set $M \subset \mathbb{R} \cup \{\infty\}$ has a smallest upper bound $\mathrm{Sup}(M)$ in $\mathbb{R} \cup \{\infty\}$. One extends the addition to $\mathbb{R} \cup \{\infty\}$ by $x + \infty = \infty + x$ for all x and similarly the multiplication with a positive $C > 0$ by $C\infty = \infty$.

A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called a *Baire function* if there exists an increasing sequence $f_n \in \mathcal{C}_c(X)$, $f_1 \leq f_2 \leq \ldots$ such that $f(x) = \mathrm{Sup}\{f_n(x);\ x \in X\}$. One can show that

$$I_B(f) := \mathrm{Sup}\{I(f_n)\}$$

is independent of the choice of the sequence. Every $f \in \mathcal{C}_c(X)$ is a Baire function and in this case $I_B(f)$ agrees with $I(f)$. We mention that the function "constant ∞" is a Baire function. Hence we can define for an arbitrary nowhere negative function $f : X \to \mathbb{R} \cup \{\infty\}$

$$\bar{I}(f) = \mathrm{Inf}\,\{I_B(h); \quad f \leq h \text{ Baire function}\}.$$

The general rule $\bar{I}(f+g) \leq \bar{I}(f) + \bar{I}(g)$ holds.

Now one can define integrable functions:

A function $f : X \to \mathbb{R}$ is called integrable if there there exists s sequence $f_n \in \mathcal{C}_c(X)$ such that $\bar{I}(|f - f_n|)$ is finite and tends to zero.

One can show that then $(I(f_n))$ converges and that the limit

$$I_L(f) = \lim_{n\to\infty} J(f_n).$$

This is called the integral of f. One can show even more that Baire functions f with finite $I_B(f)$ (for example elements of $\mathcal{C}_c(X)$) are integrable and that $I_L(f) = \lim_B(f)$ in this case. Hence we can simply write $I(f) = I_B(f)$ for Baire functions and $I(f) = I_L(f)$ for integrable functions. $I(f) = \bar{I}(f)$ for $f \in \mathcal{L}^1(X, dx)$. It is easy to see that the space $\mathcal{L}^1(X, dx)$ of all integrable functions is a vector space. It has the property that with f also $|f|$ is integrable. The integral is a linear functional on $\mathcal{L}^1(X, dx)$ with the property $I(f) \ge 0$ for $f \ge 0$.

A function $f : X \to \mathbb{C}$ is called a *zero function* if $\bar{I}(|f|) = 0$. This means that for each $\varepsilon > 0$ there exists a Baire function h with $|f| \le h$ and $I(h) < \varepsilon$. It is easy to see that zero functions are integrable. A subset of X is called a *zero subset* if its characteristic function is a zero function.e. A function f is a zero function if an only if $\{x;\ f(x) \ne 0\}$ is a zero set. If f is integrable and g is a function that coincides with f outside a zero set then g is integrable too and $I(f) = I(g)$.

We recall the basic limit theorems:

2.1 Theorem of Beppo Levi. *Assume that $f_1 \le f_2 \dots$ is an increasing sequence of integrable functions such that the sequence of their integrals is bounded. Then the pointwise limit $f(x) = \lim f_n(x)$ exists outside a zero set. If one defines $f(x)$ arbitrarily for this zero set, one gets an integrable function with the property* BepLev

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

2.2 Lebesgue's limit theorem. *Let $f_n(x)$ be a pointwise convergent sequence of integrable functions. Assume that there exists an integrable function h with the property $|f_n(x)| \le h(x)$ for all n and x. Then $f(x) = \lim f_n(x)$ is integrable and one has* LebLim

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

The subset $\mathcal{N} \subset \mathcal{L}^1(X, dx)$ of zero functions is a sub-vector space and the integral factors through the quotient

$$L^1(X, dx) := \mathcal{L}^1(X, dx)/\mathcal{N}.$$

From the limit theorems one can deduce that this space gets a Banach space with the norm

$$\|f\|_1 := \sqrt{\int_X |f(x)|dx}.$$

(Usually we will denote the class of an element $f \in L^1(X, dx)$ in $\mathcal{L}(X, dx)$ by the same letter f. A more careful notation would be to use a notation like $[f]$ for the class. For sake of simplicity we avoid this as long it is clear whether we talk of f or of its class.)

Let us assume that the Radon measure is non-trivial in the following sense: Let $f \in \mathcal{C}_c(X)$ be a non-negative function with the property $I(f) = 0$. Then $f = 0$. For such a measure the natural map

$$\mathcal{C}_c(X) \longrightarrow L^1(X, dx)$$

is injective and $L^1(X, dx)$ is the completion of $\mathcal{C}(X)$ with respect to the norm $\| \cdot \|_1$. Hence integration theory can be understood as a concrete realization of the completion.

There is another important notion:

*A function $f : X \to \mathbb{C}$ is called **measurable** if for any non-negative function $h \in \mathcal{C}_c(X)$ the function*

$$f_h(x) := \begin{cases} f(x) & \text{if } -h(x) \le f(x) \le h(x), \\ 0 & \text{else} \end{cases}$$

is integrable.

Integrable functions are measurable. All continuous functions are measurable. Measurability is conserved under all kind of standard constructions of functions which are used in analysis as addition and multiplication of functions but also taking pointwise limits and constructions as sup, inf, lim sup, lim inf for sequences of functions. A subset of X is called measurable if its characteristic function is measurable. Open subsets of X are measurable. Complements of measurable sets are measurable. Countable unions and intersections of measurable sets are measurable. Hence all sets which can be constructed from open and closed subsets be taking countable unions and intersections and complements are measurable with respect to each Radon measure. (They are called Borel sets.) So the statement *"all functions are measurable"* is not really true but nearly true. (Counter examples need sophisticated application of the axiom of choice.)

2.3 Theorem. *A function f is integrable if and only if it is measurable and if $\bar{I}(|f|) < \infty$.* InMe

Together with the previous remark this means that integrability means a kind of boundedness.

Let $p \ge 1$. The spaces $\mathcal{L}^p(X, dx)$ consist of all measurable functions f such that $|f|^p$ is integrable. This is the case for zero functions. One defines

$$\|f\|_p := \int_X \sqrt[p]{\int_X |f(x)|^p dx}.$$

This satisfies the triangle inequality. It induces a norm on the space

$$L^p(X,dx) = \mathcal{L}^p(X,dx)/\mathcal{N}$$

which is a Banach space with this norm. The case $p = 2$ is of special importance. One can consider on $\mathcal{L}^2(X,dx)$ the hermitian form

$$\langle f,g\rangle := \int_X f(x)\overline{g(x)}dx.$$

This induces a positive definite form on $L^2(X,dx)$ and equips this space with a structure as separable Hilbert space.

As a special example one can take the space $X = \mathbb{N}$ equipped with the discrete topology and the Radon measure $I(a) = \sum_n a_n$. The associated L^2-space is ℓ^2.

There is an extension of measure theory, the Bochner integral. For a Banach space E we can consider the space of compactly supported continuous functions $\mathcal{C}_c(X,E)$ with values in E.

2.4 Lemma. *Let (X,dx) be a Radon measure and E a Banach space. There exists a unique linear map* BanInt

$$\mathcal{C}_c(C,E) \longrightarrow E, \quad f \longmapsto \int_X f(x)dx,$$

such that for each continuous linear functional $L : E \to \mathbb{C}$ one has

$$L\Big(\int_X f(x)dx\Big) = \int_X L(f(x))dx.$$

The uniqueness follows directly from the Hahn-Banach theorem. So the existence, but not so quite obvious. Since for our purposes it would be sufficient to treat the case of Hilbert spaces we mention that the existence in this case is a direct consequence of the Theorem of Riesz.

There is also the notion of a measurable function. We only need it in the case where E is separable which means that it contains a countable dense subset. Then a function $f : X \to E$ is measurable if and only if its composition with all continuous linear forms is measurable. A measurable function $f : X \to E$ is called a zero function if it is zero outside a zero set. Now the spaces $\mathcal{L}^p(X,E,dx)$ can be defined in the same way as in the case $E = \mathbb{C}$. They contain the space $\mathcal{N}$ of zero functions and the quotients $L^p(X,E,dx)$ are Banach spaces. If $E = H$ is a Hilbert space, the space $L^2(X,H,dx)$ gets a Hilbert space with an obvious inner product.

Finally me mention the notion of the product measure. Let (X, dx), (Y, dy) be two locally compact spaces with Radon measures. We consider $X \times Y$ equipped with the product measure. This also locally compact space. Let $f \in \mathcal{C}_c(X \times Y)$. If we fix y we get a function $f(x, y)$ which is contained in $\mathcal{C}_c(X)$. It is easy to see that the integral $\int f(x, y)dy$ is contained in $\mathcal{C}_c(Y)$. Hence we can define the product measure

$$\int_{X\times Y} f(x,y)dxdy := \int_Y \Big[\int_X f(x,y)dx\Big]dy.$$

We claim that one can interchange the orders of integration, i.e.

$$\int_Y \Big[\int_X f(x,y)dx\Big]dy = \int_Y \Big[\int_Y f(x,y)dy\Big]dx.$$

This is trivial for splitting functions $f(x, y) = \alpha(x)\beta(y)$ and follows in general by means of the Weierstrass approximation theorem. Hence orders of integration can be interchanged. We will use this frequently.

3. Generalities about Haar measures

A topological group G is a group which carries also a topology such that the maps

$$G \times G \longrightarrow G,\ (g,h) \longmapsto gh, \quad G \longrightarrow G,\ g \longmapsto g^{-1},$$

are continuous. Here $G \times G$ has been equipped with the product topology. A *locally compact group* is a topological space whose underlying space is locally compact. We always assume that G has a countable basis of the topology.

A Haar measure on a locally compact group G is a non-zero left invariant Radon measure

$$\int_G f(x)dx = \int_G f(gx)dx \qquad (g \in G).$$

We make use of the fact that a non zero Haar measure always exists and is uniquely determined up to a constant factor.

The usual integral on $\mathbb{R}$ is a Haar measure on the additive group $\mathbb{R}$ and a Haar measure on the multiplicative group $\mathbb{R}^{\bullet}$ is given by

$$\int_{\mathbb{R}^{\bullet}} f(t)\frac{dt}{t}$$

where dt is the usual measure.

If $f \in \mathcal{C}_c(G)$ is a function with the properties $f \geq 0$ and $I(f) = 0$. Then $f = 0$. Hence we have $\mathcal{C}_c(G) \hookrightarrow L^p(G, dx)$.

Let $g \in G$. Then

$$f \longmapsto \int_G f(xg)dx$$

is also left invariant. Hence there exists a positive real number $\Delta(g) = \Delta_G(g)$ with the property

$$\int_G f(xg^{-1})dx = \Delta(g)\int_G f(x)dx.$$

The function $\Delta : G \to \mathbb{R}_{>0}$ is of course independent of the choice of dx. It is called the *modular function* of G. It is clearly a continuous homomorphism, $\Delta(gh) = \Delta(g)\Delta(h)$.

3.1 Lemma. *For every function $f \in \mathcal{L}^1(G, dx)$ the formula* InvMod

$$\int_G f(x^{-1})\Delta(x^{-1})dx = \int_G f(x)dx$$

holds.

Proof. One can check that the integral on the left hand side is a Haar measure. Hence it agrees with the right hand side up to constant a factor $C > 0$. Applying the formula twice we get $C^2 = 1$ and hence $C = 1$. □

The group G is called *unimodular* if $\Delta(g) = 1$ for all g. There are three obvious classes of unimodular groups:

1) Abelian groups are unimodular.

2) A group G is unimodular if its commutator subgroup is dense.

3) Compact groups are unimodular, more generally, for arbitrary G the restriction of Δ_G to any compact subgroup is trivial.

3) Discrete groups are unimodular.

The last statement is true since the only compact subgroup of the multiplicative group of positive reals is $\{1\}$.

We give an example of a group which is not unimodular. Let $P \subset \mathrm{SL}(2, \mathbb{R})$ be the group of all upper triangular matrices of determinant 1. Each p can be written in the form

$$p = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad (a \neq 0).$$

Moreover the map

$$\mathbb{R}^{\bullet} \times \mathbb{R} \xrightarrow{\sim} P, \quad (a, n) \longmapsto p,$$

is topological. Hence we can identify $\mathcal{C}_0(P)$ and $\mathcal{C}_c(\mathbb{R}^{\bullet} \times \mathbb{R})$.

3.2 Lemma. *Let $P \subset \mathrm{SL}(2,\mathbb{R})$ be the group of upper triangular matrices. Let da be a Haar measure on $\mathbb{R}^{\bullet}$ and dn a Haar measure on $\mathbb{R}$. Then then measure* HaarP

$$\int_P f(p)dp := \int_{\mathbb{R}^{\bullet}} \int_{\mathbb{R}} f(an)da\,dn$$

is a Haar measure. The modular function is

$$\Delta(p) = a^2.$$

(One can also write $\int\int f(an)dndn$ for the right hand side, since orders of integration can be interchanged, but $\int\int f(na)dadn$ would be false.)

Proof. The proof can be given by a simple calculation which rests on the formula

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ 0 & a^{-1} \end{pmatrix}.$$ □

We also need quotient measures. Let $H \subset G$ be a closed subgroup of a locally compact group G. Then H is also locally compact. We consider the coset space $H\backslash G$ that consists of all right cosets Hg. This is the quotient space of G by the natural action of H (multiplication from the right.) We equip it with the quotient topology with respect to the natural projection $G \to H\backslash G$. Then this projection is continuous and open. We claim that $H\backslash G$ is Hausdorff. Hausdorff means that the diagonal in $H\backslash G \times H\backslash G$ is closed. This means that its inverse image in $G \times G$ is closed. But this inverse image of H with respect to the map $G \times G \to G$, $(x,y) \mapsto xy^{-1}$.

Since $G \to H\backslash G$ is open, the space $H\backslash G$ is locally compact. There is also a natural continuous map

$$(H\backslash G) \times G \longrightarrow H\backslash G, \quad (Hg_1, g_2) \longmapsto Hg_1g_2$$

which as action from the right. A Radon measure dx on $X = H\backslash G$ is called G-invariant if

$$\int_{H\backslash G} f(xg)dx = \int_{H\backslash G} f(x)dx.$$

3.3 Proposition. *Let $H \subset G$ be a closed subgroup. Assume that $\Delta_G|H = \Delta_H$. Then there exists a non-zero invariant Radon measure dy on $H\backslash G$ and this Radon measure is unique up to a positive constant factor. It has the following property. Let dh be a right invariant Haar on H. Then* QuotMea

$$\int_G f(x)dx = \int_{H\backslash G}\left[\int_H f(hy)dh\right]dy$$

is a right invariant measure on G.

We should mention that the function $y \mapsto \int_H f(yh)dh$ can be considered as a function on $H\backslash G$. It is continuous and with compact support there.

We indicate the general proof of the existence of an invariant measure. A function $h : H\backslash G \to \mathbb{C}$ is called special if it can be written in the form $h(y) = \int_H f(hy)dh$ with a function $f \in \mathcal{C}_c(G)$. It is not difficult to show that there are many special functions such that is sufficient to define a radon measure on them. For the special h we define the integral $\int_{H\backslash G} f(y)dy$ using the formula in Proposition 3.3 where on the left hand side a Haar measure is taken. There is a problem. The function h does not determine f uniquely. Hence one has to prove a Lemma. One has to show

$$\int_H f(hy)dh = 0 \Longrightarrow \int_G f(x)dx.$$

It is a good exercise to to this for a finite group G. The integral then just are finite sums. In the general case the condition $\Delta_G|H = \Delta_H$ will play a role. In this way we get the existence of an invariant measure on $H\backslash G$ such the claimed formula holds. The proof of the uniqueness of the quotient measure is the same as the proof of the uniqueness of the Haar measure. □

We also mention that the formula in Proposition 3.3 holds for all $f \in \mathcal{L}^1(G, dx)$ with the usual caution: the inner integral exists outside of a set of measure zero and gives – extended arbitrarily – an integrable function on $H\backslash G$.

Instead of $H\backslash G$ one can also consider the space of left cosets G/H and G acts by multiplication from the left. Proposition 1.3.3 remains treu if one replaces “right” by “left”. The two versions can be transformed into each other using the transformation $g \mapsto g^{-1}$.

4. Generalities about representations

A representation π of a group G on a complex vector space is a homomorphism $\pi : G \to \mathrm{GL}(V)$ of G into the group of $\mathbb{C}$-linear automorphisms of V. Frequently we will write $g(a)$ or even simply ga instead of $\pi(g)(a)$. The map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

then has the properties:

1) $ea = a$ for all $a \in V$ (e denotes the unit element of G).
2) $(gh)a = g(ha)$ for all $g, h \in G$, $a \in V$.
3) $g(a + b) = g(a) + g(b)$, $g(Ca) = Cga$ $(C \in \mathbb{C})$.

Conversely, a map with the properties 1)-3) comes from a unique representation π.

Left and Right

Let G be a group and V simply a set. A map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

with the properties 1)-2) is also called an action of G *from the left* on V. If one replaces in 2) the condition by $(gh)a = h(g(a))$ one gets the notion of an action *from the right.* This looks better if one uses the notation ag instead of ga since then the rule takes the better looking form $a(gh) = (ag)h$. If ga is an action from the left then $g^{-1}a$ is an action from the right, and conversely. Hence there is no essential difference between the two. Keep in mind that due to our definition representations are actions from the left.

Continuous representations

There are several equivalent ways to define when a representation of a locally compact group on a Banach space is continuous. A natural way is a follows.

4.1 Definition. *A representation of a locally compact group G on a Banach space is called continuous if the corresponding map* DCr

$$G \times E \longrightarrow E$$

is continuous.

Here $G \times E$ of course carries the product topology. For a continuous representation the operators $\pi(g) : E \to E$ are continuous (hence bounded) and the map $G \to E$, $g \mapsto g(a)$, is continuous for each $a \in E$. The converse is also true.

4.2 Proposition. *A representation π of a locally compact group G on a Banach space E is continuous if all operators $\pi(g) : E \to E$ are bounded and if the map* PCr

$$G \longrightarrow E, \quad g \longmapsto \pi(g)(a),$$

is continuous for all $a \in E$.

The proof rests on the *theorem of uniform boundedness*:

4.3 Theorem. *Let E be a Banach space and let $\mathcal{M} \subset B(E)$ be a set of bounded operators such that $\{Aa,\ a \in E\}$ is bounded for each $a \in E$. Then $\mathcal{M}$ is a bounded subset of $B(E)$.* TUB

We omit the prove. □

For the proof of Proposition 4.2 we need another observation.

4.4 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a continuous representation and $K \subset G$ a compact subset. Then the set $\pi(K)$ is bounded in* $\mathrm{End}(E)$. LRepB

Proof. Since $\pi(K)a$ is compact and hence bounded for all a, the theorem of uniform boundedness gives the claim. □

Proof of Proposition 4.2. It is sufficient to prove the $\pi : G \times E \to E$ is continuous at a point (e, a). The proof follows from the lemma and the estimate

$$\|g(x) - a\| \le \|g(x) - g(a)\| + \|g(a) - a\|. \qquad \square$$

The condition of continuity in the definition of a representation can be further weakened.

4.5 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a homomorphism with the following properties:* CharCont

1) *all $\pi(g)$ are bounded.*
2) *There is a neighborhood of the identity whose image in $B(E)$ is bounded.*
3) *There is a dense subset of vectors $a \in E$ such that $g \mapsto \pi(g)(a)$ is continuous.*

Then π is a continuous representation.

Proof. We have to show that for fixed a the function $x \mapsto \pi(x)a$ is continuous. It is obviously enough to proof this at the unit element $x = e$. Hence we have to estimate $\|\pi(x)a - a\|$. For some b in the dense subset we use the estimate

$$\|\pi(x)a - a\| \le \|\pi(x)a - \pi(x)b\| + \|\pi(x)b - b\|\|b - a\|.$$

If we choose b close enough to a we obtain the desired result. □

Algebraic Irreducibility

Let $\pi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \subset V$ is called invariant if $g \in G$ and $a \in W$ implies $ga \in W$. Then we obtain a representation $\pi' : G \to \mathrm{GL}(W)$. A representation $\pi : G \to \mathrm{GL}(V)$ is called *algebraically irreducible* if $V \ne 0$ and if besides $\{0\}$ and V there are no invariant subspaces. Let W_1, W_2 be two invariant subspaces of V. Then $W_1 + W_2$ and $W_1 \cap W_2$ are also invariant. If W_1 and W_2 are irreducible then either they are equal or their intersection is zero.

Topological Irreducibility

Let now $\pi : G \to \mathrm{GL}(V)$ be a *continuous* representation. It is called *topologically irreducible* if there is no *closed* invariant subspace different from $\{0\}$ and V.

For finite dimensional representations (this means that V is finite dimensional) algebraic and topological irreducibility is the same.

A representation of a topological group on a Hilbert space H is called *unitary* if it is continuous and if all operators $\pi(g)$ are unitary operators. This means concretely

$$\langle ga, gb\rangle = \langle a, b\rangle$$

for $a, b \in H$ and $g \in G$. It is enough to demand this for $a = b$. If we talk about an irreducible unitary representation, we always mean that it is topologically irreducible.

We describe a fundamental example of a unitary representation. Let G be a locally compact group. We consider a closed subgroup $H \subset G$. For sake of simplicity we assume that both are unimodular. Then dx is left- and right invariant. We consider the space of right cosets $H\backslash G$. The group G acts on $H\backslash G$ by multiplication from the right. This is an action from the right. Let $f : H\backslash G \to \mathbb{C}$ be a function and $g \in G$. We define the translate $R_g f$ of f by $(R_g f)(x) = f(xg)$. This is an action from the left of G on the set of function on $H\backslash G$. This defines a map

$$R : G \longrightarrow \mathrm{GL}(L^2(H\backslash G, dx)).$$

By means of 4.3 one can show that this representation is continuous. It is obviously a unitary representation. In the special case $H = \{e\}$ one obtains the so-called regular representation of G on $L^2(G)$.

One of the basic problems of harmonic analysis is the investigation of this representation and to describe its spectral decomposition. This problem has been studied for the regular representation of semi simple groups G (for example $\mathrm{SL}(n, \mathbb{R})$) by Harish Chandra. In the theory of automorphic forms one studies the case where $H = \Gamma$ is a discrete subgroup such that $\Gamma\backslash G$ has finite volume.

What means "spectral decomposition"? This is not so easy to explain and not the goal of these notes. Nevertheless it is useful to get an idea of it. We give two examples. The first example is the group S^1 of complex numbers of absolute value one (circle group). The functions f on S^1 correspond to the periodic functions (period 2π) F on $\mathbb{R}$ through

$$F(t) = f(\exp(2\pi \mathrm{i} t)).$$

From the theory of Fourier series one knows that $L^2(S^1)$ is the direct Hilbert sum of the one dimensional subspaces $H(n)$ spanned by $f(\zeta) = \zeta^n$ $(n \in \mathbb{Z})$. These are invariant subspaces. The spectral decomposition of the regular representation of S^1 is

$$L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} H(n).$$

The second example deals with the regular representation of $\mathbb{R}$. There are also one dimensional spaces $H(t)$ generated by the function $x \mapsto e^{2\pi \mathrm{i} t x}$ which are invariant under translations $t \mapsto t + a$. Now t can be an arbitrary real number.

But the difference is that now $H(t)$ is not contained in $L^2(\mathbb{R})$. Nevertheless the theory of Fourier transformation shows that all f in a certain dense subspace of $L^2(\mathbb{R})$ can be written in a unique way in the form

$$f(t) = \int_{-\infty}^{\infty} g(t)e^{2\pi \mathrm{i} t}dt.$$

Hence one is tempted to say that $L^2(\mathbb{R})$ is the direct integral of the spaces $H(t)$ and to write this in the form

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} H(t)dt.$$

For general G the spectral decomposition will include both types (discrete and continuous spectra) and the constituents will not be one-dimensional but irreducible unitary representations (often infinite dimensional).

Intertwining Operators

A morphism between two continuous representations $\pi_i : G \to \mathrm{GL}(E_i)$ on Banach spaces is a continuous linear map $E_1 \to E_2$ which is compatible with the action of G in an obvious sense. Such morphisms are also called “intertwining operators”. It is clear what it means that an intertwining operator is an isomorphism. If $F \subset E$ is a closed G-invariant subspace then the natural inclusion $F \hookrightarrow E$ is a morphism. We call (G, F) a sub-representation of (G, E).

For unitary representations we will make use of a more restrictive notion of isomorphy. An isomorphism $H_1 \to H_2$ between two unitary representations $\pi : G \to \mathrm{GL}(H_i)$ is called a *unitary isomorphism,* or an isomorphism of unitary representations, if the isomorphism $H_1 \to H_2$ is an isomorphism of Hilbert spaces. This means that it preserves the scalar products.

5. The convolution algebra

let G be a locally compact group with a chosen Haar measure. The convolution of two functions $f, g \in \mathcal{C}_c(G)$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy.$$

The convolution defines an associative product on $\mathcal{C}_c(G)$. We leave the proof of the associativity as an exercise. Hence $\mathcal{C}_c(G)$ has the structure of an associative $\mathbb{C}$-algebra.

Let $\pi : G \to \mathrm{GL}(H)$ be a continuous representation on a Banach space. For any $f \in \mathcal{C}_c(G)$ and any $h \in H$ we can consider the function

$$G \longrightarrow H, \quad x \longmapsto f(x)\pi(x)h.$$

It is continuous and with compact support. Hence we can define the integral

$$\int_G f(x)\pi(x)h dx.$$

If we vary h we get an operator $H \to H$. One can check that it is linear and continuous.

We denote this operator by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

One verifies

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2).$$

What we obtain is an algebra homomorphism

$$\pi : \mathcal{C}_c(G) \longrightarrow \mathrm{End}(H).$$

The image of π consists of continuous linear operators $T : H \to H$.

Now we assume that H is a Hilbert space. We denote the adjoint of an operator $T \in \mathrm{End}(H)$ by T^*. It is defined by the formula $\langle Tx, y\rangle = \langle x, T^*y\rangle$. The existence of T follows from the Riesz lemma. Of course T^* is continuous as T, and moreover both have the same norm.

We define

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}.$$

We now assume that π is unitary. It is easy to check in this case that the new map π has the property that

$$\pi(f^*) = \pi(f)^*.$$

What we obtained is a $*$-algebra representation. We describe briefly what this means. An (associative) algebra A is a vector space (in our case over $\mathbb{C}$) together with a bilinear map

$$A \times A \longrightarrow A, \quad (a, b) \longmapsto ab.$$

We assume that this is associative but we do not assume that A contains a unit element. An involution on A is a map

$$A \longrightarrow A, \quad a \longmapsto a^*,$$

with the properties

a) $(a + b)^* = a^* + b^*, \quad (Ca)^* = \bar{C}a^*,$
b) $(ab)^* = b^*a^*.$
c) $a^{**} = a.$

5.1 Definition. *A $*$-algebra $(A, *)$ is an associative algebra (not necessarily with unit) together with a distinguished involution* DefStar

An example of a $*$-algebra is the convolution algebra $\mathcal{C}_c(G)$ with the involution defined above. Another example of a $*$-algebra is the space $\mathrm{End}(H)$ of continuous linear operators on a Hilbert space H. Multiplication is the composition of operators and the $*$-operator is given by the adjoint.

By a representation of an algebra A on a vector space V one understands a linear map $A \to \mathrm{End}(V)$ which is compatible with multiplication. By a $*$-algebra representation of a $*$-algebra A on a Hilbert space H we understand a representation

$$A \longrightarrow \mathrm{End}(H)$$

such the image of A consists of continuous operators and that is also compatible with the star operators. What we have seen that a unitary representation $\pi : G \to \mathrm{GL}(H)$ induces a $*$-algebra representation $\pi : \mathcal{C}_c(G) \to \mathrm{End}(H)$.

There are obvious notions of irreducibility:

A representation $A \to \mathrm{End}(V)$ of an algebra is called algebraically irreducible if the image of A is not zero and if there is no invariant subspace of V different form 0 and V.

A $*$-algebra representation $A \to \mathrm{End}(H)$ is called topologically irreducible if the image if A is non zero and if there is no closed invariant subspace of H different from 0 and H.

An example of a finite dimensional algebra representation is the tautological representation of $A = \mathrm{End}(V)$ on V. It is just the identity map $\mathrm{End}(V) \to \mathrm{End}(V)$. At least in the finite dimensional case it is clear that this representation is irreducible. A special case of a fundamental structure theorem of Wedderburn states (in the case of the ground field $\mathbb{C}$):

5.2 Theorem. *Let $\pi : A \to \mathrm{End}(V)$ be an irreducible representation of an algebra A on a finite dimensional vector space V. Then π is surjective.* WEDD

We don't give the proof here and refer to the text book of S. Lang on algebra. To be honest, we mention that Lang treats only the case where A contains a unit element. The general case can be reduced by the technique of adjoining a unit element.

A trivial consequence of Theorem 5.2 is as follows. Let $T : V \to V$ be a linear operator that commutes with all $\pi(a)$, $a \in A$. Then T is a multiple of the identity. A basic result states that this carries over to the infinite dimensional case.

5.3 Theorem (Schur's lemma for algebra representations). *Let π be a topologically irreducible unitary representation of a $*$-algebra A on a Hilbert space H. Assume that $T : H \to H$ is a linear and continuous operator that commutes with all $\pi(a)$, $a \in A$. Then T is a constant multiple of the identity.* SchurL

Corollary. *If A is abelian then then H is one-dimensional.*

We will not give the proof in the infinite dimensional case in this text. We just mention that it rests on the spectral theorem for normal operators. □

The same theorem is true for irreducible unitary representations of locally compact groups. Actually it is a consequence of Theorem 5.3 as we shall point out. The argument would be very easy if there exists for $g \in G$ a Dirac function $\delta_g \in \mathcal{C}_c(G)$ which means

$$\delta_g(x) = 0 \text{ for } x \neq g \quad \text{and} \quad \int_G \delta_g(x)dx = 1.$$

Such a situation is of course rare, but it occurs, namely for finite groups. A simple computation then gives $\pi(\delta_g) = \pi(g)$. From this one can deduce that a subspace of H is invariant under all $\pi(g)$, $g \in G$, if and only if it is invariant under all $\pi(f)$, $f \in \mathcal{C}_c(G)$. Actually there is a weak variant of Dirac functions.

5.4 Lemma. *For each locally compact group G there exists a sequence of functions $\delta_n \in \mathcal{C}_c(G)$ with the following properties.* DirSeq

1) $\mathrm{supp}(\delta_{n+1}) \subset \mathrm{supp}(\delta_n)$.
2) *For each neighborhood U of the identity there exists an n such that* $\mathrm{supp}(\delta_n) \subset U$.
3) $\delta_n(x^{-1}) = \delta_n(x)$.
4) $\delta_n(x) \geq 0$ *and* $\int_G \delta_n(x)dx = 1$.

We call (δ_n) a *Dirac sequence.*

5.5 Lemma. *Let (δ_n) be a Dirac sequence. Then $\pi(\delta_n)$ converges to the identity in the sense* DiracLim

$$\lim_{n\to\infty} \|\pi(\delta_n)h - h\| = 0.$$

(This means pointwise convergence.)

Proof. We have

$$\|\pi(\delta_n)h - h\| \leq \int_G \delta_n(x)\|\pi(x)h - h\|.$$

Let $\varepsilon > 0$. For n big enough we have $\|\pi(x)h - h\| < \varepsilon$ for all $x \in U_n$. We obtain $\|\pi(\delta_n)h - h\| < \varepsilon$. □

There is an obvious generalization. Let $g \in G$ then from Lemma 5.5 we see that $\pi(f_n) \circ \pi(g) \to \pi(g)$ (pointwise) A simple calculation shows

$$\pi(f) \circ \pi(g) = \pi(\tilde{f}) \quad \text{where} \quad \tilde{f}(x) = \Delta(g)f(xg^{-1}).$$

This shows the following result.

5.6 Lemma. *Let $G \to \mathrm{GL}(H)$ be a unitary representation and let $W \subset H$ be a closed subspace. Assume that there exists a subalgebra $A \subset \mathcal{C}_c(G)$ that contains a Dirac sequence and that is invariant under translation $f(x) \mapsto f(xg)$ for all $g \in G$ and such that W is invariant under A. Then W is invariant under G.* InvDS

As an application of Lemma 5.6 we get the following lemma.

5.7 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. A closed subspace $W \subset H$ is invariant under G if and only if it is invariant under $\mathcal{C}_c(G)$.* InvDC

Schur's lemma now can be formulated also for group representations.

5.8 Theorem (Schur's lemma for group representations). *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representation of a locally compact group. Every linear and continuous operator $T : H \to H$ which commutes with all $\pi(g)$, $g \in G$, is a multiple of the identity.* SchurG

Corollary. *If G is abelian then then H is one-dimensional.*

Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. We say that another unitary representation of G occurs in π if it is isomorphic (in the unitary sense) to a sub-representation of π.

5.9 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representations and A, B be two invariant closed subspaces. Assume the the restriction of π to A is (topologically) irreducible. Then either A is orthogonal to B or the representation $\pi|A$ occurs in $\pi|B$.* IrrOrth

Corollary. *If both A and B are irreducible then either they are orthogonal or isomorphic (as G-representations).*

Proof. We consider the pairing $\langle\cdot,\cdot\rangle : A \times B \to \mathbb{C}$. We first notice that it is non degenerate in the following sense. For each $a \in A$ their exists a $b \in B$ such that $\langle a, b\rangle \neq 0$ and conversely. This is clear since the orthogonal complement of B intersected with A is a closed invariant subspace. Next we construct a linear map $f : A \to B$. By the Lemma of Riesz there exists for each $a \in A$ a unique $f(a)$ in B such that $\langle a, b\rangle = \langle f(a), b\rangle$ for all $b \in B$. One easily checks that this is an intertwining operator. □

5.10 Definition. *A unitary representation $\pi : G \to \mathrm{GL}(H)$ is called* ***completely reducible*** *if H can be written as the direct Hilbert sum of pairwise orthogonal closed invariant subspaces* DefCI

$$H = \widehat{\bigoplus_i} H_i$$

which are irreducible as G-representations.

In general we denote by $\hat{G}$ the set of all isomorphy classes of irreducible unitary representations of G and call it the *unitary dual* of G. Recall that each irreducible unitary representation $\pi : G \to \mathrm{GL}(H)$ is one dimensional if G is abelian. Hence it is of the form $\pi(g)(h) = \chi(g)h$ where χ is a character of G. By definition, this is a continuous homomorphism from G into the group of complex numbers of absolute value 1. Unitary isomorphic representations give the same character. This gives a bijection with $\hat{G}$ and the set of all unitary characters. Characters can be multiplied in an obvious way. Hence, for abelian G, the set $\hat{G}$ is a group as well. One can show that it carries a structure as locally compact group.

5.11 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ a unitary representation which is completely reducible,* EindIsot

$$H = \widehat{\bigoplus_{i\in I}} H_i, \qquad H_i \subset H.$$

Let $\tau \in \hat{G}$. Then

$$H(\tau) = \widehat{\bigoplus_{i\in I,\ \pi_i \in \tau}} H_i$$

is the closure of the sum of all irreducible closed invariant subspaces of H that are of type τ. In particular, it is independent of the choice of the decomposition.

This follows immediately from Lemma 5.9. □

We call $H(\tau)$ the τ-isoptypic component of π. This is well-defined. The irreducible components H_i are usually not well-defined. Look at the example of the group G that consists only of the unit element. Nevertheless the so-called multiplicity

$$m(\tau) := \#\{i \in I;\ \pi_i \in \tau\} \le \infty$$

is independent on the choice of the decomposition. This can be seen as follows. Let $(H(\tau), \tau)$ be a realization of τ. We consider the vector space of all intertwining operators $H \to H(\tau)$. The space of intertwining operators $H_i \to H(\tau)$ is zero if π_i is not in τ and – by Schurs's lemma – one dimensional otherwise. From this follows easily the space of intertwining operators $H \to H(\tau)$ has dimension $m(\tau)$. This shows the invariance of $m(\tau)$.

This gives us the following result.

5.12 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a completely reducible unitary representations. The multiplicities* MultIso

$$m(\tau) := \#\{i \in I;\ \pi_i \in \tau\} \le \infty$$

(in the notation of Proposition 1.5.11 are well-defined). Two completely reducible representations are unitary isomorphic if and only of their multiplicities agree.

6. Generalities about compact groups

In this section we treat some general facts about representations of compact groups. Readers who are mainly interested in the classification of the irreducible unitary representations of the group $\mathrm{SL}(2,\mathbb{R})$ can skip this section, since the only compact group which occurs in this context is the group $\mathrm{SO}(2,\mathbb{R})$. This group is not only compact but also abelian which makes the theory rather trivial.

We need some results of functional analysis. We recall the notion of equicontinuity:

6.1 Definition. *A set $\mathcal{M}$ of functions on a topological space X is called equicontinuous at a point $a \in X$ if for any point $\varepsilon > 0$ their exists a neighborhood U of a such that* EquiCon

$$|f(x) - f(a)| < \varepsilon \quad \textit{for all} \quad x \in U,\ f \in \mathcal{M}.$$

The set is called equicontinuous if this is the case at all $a \in X$.

(The point is the independence of the neighborhood U from f.) We recall a basic result from functional analysis.

6.2 Theorem (theorem of Arzela-Ascoli). *Let X be a locally compact space with countable basis of the topology. Let $\mathcal{M}$ be an equicontinuous set of functions on X such that the set of numbers $f(x)$, $f \in \mathcal{M}$, is bounded for every $x \in X$. Then each sequence of $\mathcal{M}$ admits a subsequence that converges locally uniformly on X.* AA

There are variants of this theorem in which equicontinuity does not appear. Let for example $X \subset \mathbb{R}^n$ be an open subset and assume that $\mathcal{M}$ is a set of *differentiable* functions such that there exists a constant C such that

$$|f(x)| \le C \quad \text{and} \quad |(\partial f/\partial x)(x)| \le C \quad \text{for all } x \in X.$$

Then the mean value theorem of calculus shows that this set is equicontinuous.

Another main tool will be the spectral theorem for compact operators on Hilbert spaces. Let H be a Hilbert space. A linear and continuous operator $T : H \to H$ is called *compact* if the image any bounded set is contained in a compact set. For example this is the case if the image of T is finite dimensional. The identity is compact if and only if H is finite dimensional. The set of all compact operators is closed under the operator norm. So, let $T_1, T_2, \ldots$ be a sequence of compact operator and T another bounded operator such that $\|T_n - T\|$ tends to 0. Then T is compact.

Recall that an operator T is called normal if it commutes with its adjoint, $T \circ T^* = T^* \circ T$.

6.3 Theorem (Spectral theorem for compact operators). *Let $T : H \to H$ be a compact and normal operator. The set of eigenvalues is either finite or it is countable and 0 is the only accumulation point of it. The eigenspaces $H(T,\lambda)$ are pairwise orthogonal and for $\lambda \neq 0$ they are finite dimensional. The sum of all eigenspaces is dense in H. Hence we have a Hilbert space decomposition* KompOp

$$H = \widehat{\bigoplus}_{\lambda} H(T,\lambda).$$

We will not proof this theorem here.

We give an example of a compact operator.

6.4 Proposition. *Let X be a compact topological space and dx a Radon measure. Let $K \in \mathcal{C}(X,X)$ be a continuous function. The operator* IntOp

$$L_K : L^2(X,dx) \longrightarrow L^2(X,dx), \quad L_K(f)(x) := \int_X K(x,y)f(y)dy.$$

is a compact (continuous and linear) operator.

We mention that every square integrable function f on a compact space is integrable (since one can write $f = 1 \cdot f$ as product of two square integrable functions). Since $K(x,y)$ for fixed x is an L^2-function the existence of the integral in Proposition 6.4 is clear. Clearly the functions $L_K f$ are continuous. Even more we have

$$|L_X(f)(x)| \leq c\|f\|_2$$

with some constant c by the Cauchy-Schwarz inequality. This also implies that $L_X f \in L^2(X,dx)$ and moreover

$$\|L_K f\|_2 \leq C\|f\|_2$$

with some constant C. Hence the operator is linear and also continuous.

But we have a stronger property. It is easy to show that the set of functions

$$\{L_K f; \quad f \in L^2(X,dx), \quad \|f\|_2 \leq 1\}$$

is equicontinuous. This implies that L_K is a compact operator. For this we have to prove the following. Let $f_n \in L^2(X,dx)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$. We have to show that $L_K f_n$ has a sub-sequence that converges in $L^2(X,dx)$. The theorem of Arzela-Ascoli shows that $L_K f_n$ converges uniformly. Hence it converges point-wise and and all functions are bounded by a joint constant. Since X is compact, constant functions are integrable and we can apply the Lebesgue limit theorem to obtain convergence in $L^2(X,dx)$. □

6.5 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation of a locally compact group G on a Hilbert space G. Assume that there exists a Dirac sequence $\delta_n \in \mathcal{C}_c(G)$ such that all $\pi(\delta_n)$ are compact operators. Then the representation decomposes into irreducibles with finite multiplicities.* CompRed

Proof. We consider pairs that consist of a closed invariant subspace $H' \subset H$ such the restriction of π to H' is completely reducible and a distinguished decomposition $H' = \widehat{\bigoplus}_{i \in I} H'_i$ into irreducibles. We define an ordering for such pairs. The pair $H' = \widehat{\bigoplus}_{i \in I} H'_i$ is less or equal than the pair $H'' = \widehat{\bigoplus}_{j \in J} H''_j$ if each space H'_i equals some H''_j. (Especially $H' \subset H''$). From Zorn's lemma easily follows that there exists a maximal member. We call its orthogonal complement U. This space cannot contain any irreducible subspace since this could be used to enlarge the maximal element. Hence we have to show:

let π be a representation as in the proposition which is not zero. Then there exists at least one irreducible closed subspace.

To prove this we choose an element f of the Dirac sequence such that $\pi(f)$ is not identically zero. This element will kept fixed during the proof. We also choose an eigenvalue $\lambda \neq 0$ of $\pi(f)$ Let $H(f, \lambda) \subset H$ the eigenspace. This is a finite dimensional vector space.

There may be other invariant closed subspaces which have a non-zero intersection with $H(f, \lambda)$. We choose a a closed subspace E such that the dimension of its intersection with $H(f, \lambda)$ is non-zero and minimal. Then we set $W = E \cap H(f, \lambda)$. There still may exist several closed invariant subspaces H that share with E the property $W = F \cap H(f, \lambda)$. We take the intersection of all these F and get in this way a smallest closed invariant subspace $F \subset E$ with $W = F \cap H(f, \lambda)$. We claim that this F is irreducible. For this we take any orthogonal decomposition $F = A \oplus B$. The eigenvalue λ must occur as eigenvalue of λ in one of the spaces A, B. (The restriction of a compact operator to a closed invariant subspace remains compact and hence decomposes into eigen spaces.) Let us assume that it occurs in A. Then $A \cap H(f, \lambda)$ is not zero. It must agree with W because of the minimality property of $\dim W$. Moreover it must agree with F because of the minimality property of F. This shows the irreducibility.

It remains to prove that the multiplicities are finite. Let $\tau \in \hat{G}$. Let $H_1, \dots, H_m$ be pairwise orthogonal invariant closed subspaces of type τ. We claim that m is bounded. There exists an element $f = \delta_n$ from the Dirac sequence such that $\pi(f)$ is not zero on H_1. There exists a non-zero eigenvalue λ. This eigenvalue then occurs in all H_i since they are all isomorphic (as representations). Since the multiplicity of the eigenvalue is finite the number m must be bounded. □

A special case of Proposition 6.5 gives the following basic result.

6.6 Theorem. *Let K be a compact group. The regular representation of* CompIrr

K on $L^2(K)$ (translation from the right) is completely reducible with finite multiplicities.

Proof. Let $f \in \mathcal{C}(K)$. We have to show that the operator R_f is compact. Recall that R_f is defined as Bochner integral

$$R_f(h) = \int_K f(x)R_x(h)dx, \qquad (R_xh)(y) = h(yx).$$

It looks natural to get this as function by interchanging the evaluation if this function with integration, i.e. one should expect

$$R_f(h)(y) = \int_K f(x)h(yx)dx.$$

This is actually true but one has to be careful with the argument since the evaluation map $h \mapsto h(y)$ is not a continuous linear functional on the Hilbert space $L^2(K)$. Instead of this one uses the following argument. Two elements of a Hilbert space are equal if and only if their scalar products with an arbitrary vector are equal. Taking scalar product with a vector is a continuous linear functional which can be exchanged with the Bochner integral. In this way one obtains the desired formula. We can rewrite the formula as

$$R_f(h)(x) = \int_K f(x^{-1}y)h(y)dy.$$

This is the integral operator with kernel $K(x,y) = f(x^{-1}y)$. □

We mention two other basic result for compact groups.

6.7 Proposition. *Let $\pi : K \to \mathrm{GL}(H)$ be a Banach representation of a compact group on a Hilbert space H. There exists a hermitian product on H whose norm is equivalent to the original one and such that π is unitary.* UNitC

The proof is easy. One replaces the original hermitian product $\langle\cdot,\cdot\rangle$ by the new scalar product

$$\int_K \langle \pi(k)(x), \pi(k)(y)\rangle.$$

(This is called Weyl's unitary trick.)

6.8 Theorem. *Every topologically irreducible representation of a compact group on a Hilbert space is finite dimensional* IrCoFi

We don't give the proof here.

Theorem 6.6 admits the following generalization:

6.9 Theorem. *Let G be a unimodular locally compact group and $\Gamma \subset G$ a discrete subgroup such that $\Gamma\backslash G$ is compact. Then the representation of G on $L^2(\Gamma\backslash G)$ (translation from the right) is completely irreducible with finite multiplicities.* GGfin

Proof. As in the proof of Theorem 6.6 we can rewrite the operator R_f as an integral operator

$$\int_G f(y)h(xy)dy = \int_G f(x^{-1}y)h(y)dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)dy.$$

This is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

Since f has compact support, this sum is locally finite and K is a continuous function on $X \times X$ where X is the compact space $\Gamma\backslash G$. So we can apply Proposition 6.4. □

This theorem is of great importance for the theory of automorphic forms and is one reason to study the irreducible irreducible representations of G.

7. Some examples of unitary representations

We construct the basic unitary representations of $G = \mathrm{SL}(2,\mathbb{R})$.

The principal series

We take $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2,\mathbb{R})$. We also consider the group P of upper triangular matrices

$$p = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$$

with positive diagonal elements. Then we consider the space of all functions

$$f(py) = a^{1+s} f(y), \qquad p \in P,\ y \in G \quad (s \in \mathrm{i}\mathbb{R}).$$

The group G acts on the set of these functions by translation from the right. The Iwasawa decomposition shows that such a function is determined by its restriction to K. Hence we described an action of G on functions on K. Assume that the restriction of such a function f to K is a zero function (with respect to the Haar measure on K). It is easy to see that then the G-translates of f have the same property. We denote by $\mathcal{H}(s)$ the space of all functions on on G with the above transformation property and such that the restriction is contained in $\mathcal{L}^2(K, dk)$. We transport the hermitean product of $L^2(K, dk)$ to $\mathcal{H}(s)$. We also can consider the subspace of all functions in $\mathcal{H}(s)$ whose restriction to K is a zero function. We denote by $H(s)$ the quotient of $\mathcal{H}(s)$ by this subspace. Then $H(s)$ gets a Hilbert space which can by identified with $L^2(K, dk)$.

Let $f \in \mathcal{H}(s)$. We claim that the translates of f under G are also contained in $\mathcal{H}(s)$.

We have to use the modular function of P which is $\Delta(p) = a^2$.

7.1 Lemma. *Let $y \in G$. We consider the (continuous) maps $\alpha : K \to K$ and $\beta : K \to P$ which are defined by $ky = \beta(k)\alpha(k)$. The for each $f \in \mathcal{C}_c(K)$ the formula* TmI

$$\int_K f(\alpha(k))\Delta(\beta(k))dk = \int_K f(k)dk$$

holds.

Proof. Since G and K are unimodular we can consider on G/K the invariant quotient measure and this is a Haar measure on P which we can identify with G/K. We choose an arbitrary function $\varphi \in \mathcal{C}_c(P)$ with the property

$$\int_P \varphi(p)dp = 1.$$

Then we consider the function $F(pk) = \varphi(p)f(k)$. This is a function in $\mathcal{C}_c(G)$. The defining formula for the quotient measure on $K\backslash G$ is

$$\int_G F(x)dx = \int_P \int_K F(pk)dkdp = \int_K f(k)dk.$$

We use the right invariance of the Haar measure on G to obtain

$$\int_K f(k)dk = \int_G F(xy)dx = \int_P \int_K F(p\beta(k)\alpha(k))dkdp.$$

We first integrate over p. Since the factor $\beta(k) \in P$ os on the right from p we get

$$\int_K f(k)dk = \int_K \int_P F(p\alpha(k))\Delta(\beta(k))dkdp = \int_K f(\alpha(k)\Delta(\beta(k))dk. \qquad \square$$

To understand the meaning of Lemma 7.1 we mention that there is a natural bijection $P\backslash G \to K$. Hence we get an action of G on K from the right. One might think to consider an invariant quotient measure on $P\backslash G$. This would give a G-invariant measure on K which of course would by a Haar measure on K. But this does not work since the existence of the quotient measure needs the assumption $\Delta_G|P = \Delta_P$ which is not satisfied in our case. Lemma 7.1 actually shows how the Haar measure on K transforms under the action of G.

Lemma 7.1 also shows how the norm

$$\|f\|_2 = \int_K |f(k)|^2 dk$$

changes under the action of G. In the case that s is purely imaginary, we get $|a|^s = 1$. The remaining factor a comes out as a^2 and cancels against The Δ-factor in the transformation formula. This shows that we get an unitary representation. This series of representations is called the principal series.

7.2 Proposition (Principal Series Representations). *For each $s \in \mathrm{i}\mathbb{R}$ there is a unitary representation of $G = \mathrm{SL}(2,\mathbb{R})$ on the space $L^2(K, dk)$ which can be defined as follows. Take a square integrable function f on K and extend it to a function on G with the property* PrinS

$$f(px) = a^{1+s} f(x) \qquad (x \in G).$$

Consider the translation of G from the right on these functions. This induces an irreducible unitary representation on $H(s) = L^2(K, dk)$.

Additional Remark. *For arbitrary complex s we still get a Banach representations whose restriction to K is unitary.*

We will see that the these unitary representations play a fundamental role. They are not irreducible, but, as we will see they decompose into a few irreducible pieces.

We finally mention that the principal series representation can be defined for arbitrary complex numbers. But then it is only a Banach representation (but on a Hilbert space $H(s)$). The restriction to K is still unitary.

The discrete series

We denote by $\mathbb{H}$ the upper half plane in the complex plane. Recall that the group $G = \mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ through $(az+b)(cz+d)^{-1}$. The measure $dxdy/y^2$ is invariant under the action of G. We consider more generally for integers n the measures

$$d\omega_n = y^n \frac{dxdy}{y^2}.$$

We assume $n \geq 2$ Then we consider the space

$$H = L^2_{\mathrm{hol}}(\mathcal{H}, \omega_n)$$

of all *holomorphic* functions which are square integrable with respect to this measure. It can be shown that it is a closed subspace of $L^2(\mathcal{H}, \omega_n)$ and hence a Hilbert space. It can be shown that it is infinite dimensional We define an action π_n of $G = \mathrm{SL}(2,\mathbb{R})$ on function on $\mathbb{H}$ by means of the formula

$$(\pi_n(g)f)(z) = f(g^{-1}z)(cz+d)^{-n}.$$

It can be checked that defines a unitary representation of G. It is called the holomorphic discrete series. If one considers antiholomorphic instead of holomorphic functions one obtains the antiholomorphic discrete series.

The complementary series

We only give a very short description without details. Let be $s \in (0,1)$. It can be shown that for each $f \in \mathcal{C}_c(\mathbb{R} \times \mathbb{R})$ the function $f(x,y)/|x-y|^{s-1}$ is Lebesgue integrable on $\mathbb{R}^2$ in the usual sense. (It is defined outside the diagonal $x = y$. But since the diagonal is a zero set, this does not matter. For example we can extend by 0 to the diagonal). We denote by $\mathcal{H}^e(s)$ the space of all measurable functions f on $\mathbb{R}$ (with respect to the usual Lebesgue measure) such that $f(x)\overline{f(y)}/|x-y|^{s-1}$ is Lebesgue integrable on $\mathbb{R} \times \mathbb{R}$. So this space contains $\mathcal{C}_c(\mathbb{R})$. It can be shown that for two functions $f, g \in \mathcal{H}^e(s)$ the integral

$$\langle f, g \rangle = \int_{\mathbb{R}\times\mathbb{R}} \frac{f(x)\overline{g(y)}}{|x-y|^s} dxdy$$

exists and defines a hermitian form on $\mathcal{H}^e(s)$. We quotient out zero functions to obtain the spaces $\mathcal{H}^e(s)$. The hermitian form factors through this space. It can be shown that this gives a Hilbert space. The group $G = \mathrm{SL}(2,\mathbb{R})$ acts on this space through

$$(\pi_s^e(g)f)(x) = |cx+d|^{s-1} f(g^{-1}x).$$

One can check that this defines a unitary representation.

Chapter II. The real special linear group of degree two

1. The simplest compact group

We study the group

$$K = \mathrm{SO}(2, \mathbb{R}).$$

So K consists of all real 2×2 matrices k of determinant 1 with the property

$$k'k = e.$$

Because of

$$k^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \left(k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

this means that k is of the form

$$k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^2 + b^2 = 1.$$

For $k \in K$ the complex number $\zeta = a + \mathrm{i}b$ is of absolute value 1. Recall that the set of all complex numbers of absolute value 1 is a group under multiplication. One easily checks that the map

$$\mathrm{SO}(2, \mathbb{R}) \xrightarrow{\sim} S^1, \quad k \longmapsto \zeta,$$

is an isomorphisms of locally compact groups. So we see that K is a compact and abelian group. Hence we know that each irreducible unitary representation is one-dimensional and corresponds to a character. The characters of S^1 are easy. The correspond to the integers $\mathbb{Z}$. For each integer n we can define

$$\chi_n(k) = \chi_n(\zeta) := \zeta^n.$$

For an arbitrary unitary representation $\pi : K \to \mathrm{GL}(H)$ we can consider the corresponding isotypic component

$$H(n) := \{h \in H; \quad \pi(g)(h) = \chi_n(g)h\}.$$

Another way to write the elements of $\mathrm{SL}(2, \mathbb{R})$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Here ϕ is determined mod $2\pi\mathrm{i}\mathbb{Z}$. The character χ_n in this presentation is given by

$$\chi_n(k) = e^{2\pi\mathrm{i}n\theta}.$$

2. The Haar measure of the real special linear group of degree two

We use the following notations:

$$\begin{aligned}
&G = \mathrm{SL}(2,\mathbb{R}),\\
&A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; \quad t \in \mathbb{R} \right\},\\
&N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; \quad x \in \mathbb{R} \right\},\\
&K = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad \theta \in \mathbb{R} \right\}.
\end{aligned}$$

2.1 Lemma (Iwasawa decomposition). *The map* IwaZ

$$A \times N \times K \longrightarrow G, \quad (a,n,k) \longmapsto ank,$$

is topological.

Proof. The elements of K act as rotations on $\mathbb{R}^2$. To any $g \in G$ one can find a rotation k such that gk fixes the x-axis. Then gk is triangular matrix. This gives the prove of the lemma. □

One can write the decomposition explicitly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \frac{1}{\sqrt{c^2+d^2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & ac+bd \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

We denote the Haar measures on A, N, K by da, dn, dk. Since $t \mapsto a_t$ is an isomorphism of groups, we have $da = dt$ where dt denotes the standard measure of $\mathbb{R}$. The measure dk is normalized such that the volume of K is 1.

We first consider the group $P = AN$ of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ with positive diagonal. The map $A \times N \to P$ is topological (but not a group isomorphism). Recall that

$$\int_P f(p)dp := \int_A \int_N f(an)dnda$$

is a Haar measure (Lemma I.3.2).

2.2 Proposition. *A Haar measure on $G = \mathrm{SL}(2,\mathbb{R})$ can be obtained as follows* HaarSR

$$\int_G f(x)dx = \int_A \int_N \int_K f(ank)\, dk\, dn\, da.$$

Proof. Since K is compact we have $\Delta_G|K = \Delta_K$. Hence the invariant quotient measure On $K\backslash G$ exists. There is a natural topological map $P \to K\backslash G$. The quotient measure gives a Haar measure on P. The rest comes from defining properties of a quotient measure (Proposition 1.3.3). □

3. The space $S_{m,n}$

We consider the groups

$$G = \mathrm{SL}(2,\mathbb{R}) \quad \text{and} \quad K = \mathrm{SO}(2,\mathbb{R}).$$

Making use of the Iwasawa decomposition, we can write any function $f : G \to \mathbb{R}$ as functions of the variables a, n, θ

$$f(g) = g(a,n,\theta).$$

Since g can be considered as a function on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$, it makes sense to talk about differentiable g and in this way of differentiable f. We denote the subspace of differentiable functions of $\mathcal{C}_c(G)$ by $\mathcal{C}_c^\infty(G)$.

3.1 Definition. *The space $S_{m,n}$ consists of all $f \in \mathcal{C}_c^\infty(G)$ with the property* Smn

$$f(k_\theta x k_{\theta'}) = f(x)e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'} \qquad (x \in G).$$

Let be $f \in \mathcal{C}_c^\infty(G)$. Then the Fourier coefficient

$$f_{m,n}(x) = \int_0^{2\pi}\int_0^{2\pi} f(k_\theta x k_{\theta'})e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'}\,d\theta d\theta'$$

is contained in $S_{m,n}$. From the theory of Fourier series we obtain

$$f(x) = \sum_{m,n} f_{m,n}(x)e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'}$$

where the convergence is absolute and locally uniform in x. Let $\mathrm{supp}(f)$ be the support of f. Then $K\,\mathrm{supp}(f)K$ contains the support of $f_{m,n}$. We have proved the following result:

3.2 Lemma. *Let be $f \in \mathcal{C}_c^\infty(G)$ and let be $\varepsilon > 0$. There exists a function g which is a finite linear combination from functions contained in $S_{m,n}$ and with the following property:* LemSmn

a) $\mathrm{supp}(g) \subset K\,\mathrm{supp}(f)K$,
b) $|f(x) - g(x)| < \varepsilon$ *for* $x \in G$.

Corollary. *The algebraic sum $\sum_{m,n} S_{m,n}$ is dense in the space $L^1(G,dx)$ with respect to the norm $\|\cdot\|_1$.*

Here dx of course is a Haar measure. Recall that G is a unimodular group, hence we have to define

$$f^*(x) = \overline{f(x^{-1})}.$$

We study the convolution.

3.3 Lemma. *We have* FaltSmn

a) $S_{m,n} * S_{p,q} = 0$ *if* $n \neq p$.
b) $S^*_{m,n} = S_{n,m}$.
c) $S_{m,n} * S_{n,q} \subset S_{m,q}$.

The proof can be given by an easy calculation. We restrict to the case a). In the convolution integral

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

we replace y by yk_θ which doesn't change the integral. Now we use the transformation properties of f and g and obtain that $(f * g)(x)$ remains unchanged if one multipies it by $e^{2\pi(p-n)\theta}$, This proves a). □

From Lemma 3.3 we see that $S_{n,n}$ is a star algebra.

3.4 Proposition. *The algebra $S_{n,n}$ is commutative.* SmnCom

Proof. There is a very general principle behind this statement. It depends on the fact that $G = \mathrm{SL}(2,\mathbb{R})$ admits two involutions

$$x^\tau = x' \quad \text{(transpose of } x\text{)}$$
$$x^\sigma = \gamma x \gamma \quad \text{where} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We collect the properties of the two involutions that are needed in the proof.

1) σ is an automorphism ($(xy)^\sigma = x^\sigma y^\sigma$) and τ is an anti-automorphism ($(xy)^\sigma = y^\sigma x^\sigma$)
2) $k^\tau = k^\sigma = k^{-1}$ for $k \in K$.
3) Every element of G can be written as product sk of a symmetric matrix ($s = s^\tau$) and an element $k \in K$.
4) For every symmetric $s = s^\tau$ there exist $k \in K$ such that

$$s^\sigma = ksk^{-1}$$

for all symmetric $s = s^\tau$.

1) and 2) are clear. To prove 4) we use that any real symmetric matrix s can be transformed by means of an orthogonal matrix into a diagonal matrix

$$k_1 s k_1^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Here λ_1, λ_2 are the eigen values of s. Since we can replace k_1 by γk_1 we can assume that the determinant of k_1 is 1. The matrix s^σ is also symmetric and and has the same eigen values as s. Hence we find an orthogonal matrix k_2 of determinant 1 such that

$$k_2 s^\sigma k_2^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We obtain $ksk^{-1} = k^\sigma$ where $k = k_2^{-1} k_1$. Finally we prove 2). So, let $x \in \mathrm{SL}(2, \mathbb{R})$. We consider xx'. This is a symmetric positive definite matrix. Transformation to a diagonal matrix by means of an orthogonal matrix gives a symmetric positive matrix s with the property $xx' = s^2$. Then $k = s^{-1}x$ is orthogonal and has the desired property. This finishes the proof of 1)-4).

We also mention that the Haar measure on G is invariant under the two involutions. We give the argument for the anti-automorphism σ. The integral $\int_G f(x^\sigma)dx$ is right invariant. Since G is unimodular it agrees with $\int_G f(x)dx$ up to a positive constant factor C. Since σ is involutive we get $C^2 = 1$ and hence $C = 1$.

Now we can give the proof of Proposition 2.3.4. We extend the involutions to functions on G by

$$f^\sigma(x) = f(x^\sigma), \quad f^\tau(x) = f(x^\tau).$$

We claim the following two formulae:

$$(f * g)^\sigma = f^\sigma * g^\sigma, \quad (f * g)^\tau = g^\tau * f^\tau.$$

We prove the second formula (the first one is similar). We have

$$(f * g)^\tau(x) = \int_G f(y) g(y^{-1}x^\tau)dy \quad \text{and} \quad (g^\tau * f^\tau)(x) = \int_G g(y^\tau) f((y^{-1})^\tau x)dy.$$

In the first integral we replace y by y^τ, then y by xy and after that y by y^{-1}. This transformations don't change the integrals and proves the claimed identity.

Now we assume that $f \in S_{m,m}$. In this case we claim $f^\tau = f^\sigma$. To prove this we write $x \in G$ in the form $x = sk$. Then we get

$$f^\tau(x) = f(k^\tau s) = \varrho(k) f(s) \qquad (\varrho(k) = e^{im\theta})$$

and

$$f^\sigma(x) = f(s^\sigma k^{-1}) = \varrho(k) f(s^\sigma) = \varrho(k) f(\gamma s \gamma^{-1}) = \varrho(k) f(s).$$

Now let f, g be both in $S_{m,m}$. Then $f * g$ is in $S_{m,m}$ too and we get $(f * g)^\tau = (f * g)^\sigma$. This gives

$$g^\tau * f^\tau = f^\sigma * g^\sigma.$$

In this formula we can replace τ by σ. Since $f, g \in S_{m,m}$ implies that $f^\sigma, g^\sigma \in S_{m,m}$ we can replace f, g by f^σ, g^σ to obtain the final formula $f * g = g * f$. □

Now we consider a Banach representation of $G = \mathrm{SL}(2, \mathbb{R})$,

$$\pi : G \longrightarrow \mathrm{GL}(H).$$

We assume that H is a Hilbert space. But it is not necessary to assume that ir is unitary. We restrict this representation to K. Without loss of generality we can assume that the restriction to K is unitary (use Proposition I.6.7). We consider the (closed) subspace

$$H(m) := \{h \in H; \quad \pi(k_\theta)(h) = e^{\mathrm{i}m\theta} h\}.$$

The spaces $H(n)$ are pairwise orthogonal and that H is the direct Hilbert sum of the $H(n)$. For an element h in the algebraic sum, we denote by h_n the component in $H(n)$.

3.5 Lemma. *The space $S_{m,n}$ maps $H(n)$ into $H(m)$. It maps $H(q)$ to zero if $n \neq q$.* AHact

The proof is very easy and can be omitted. □

3.6 Lemma. *The elements of $H(m)$ are differentiable if $H(m)$ is finite dimensional.* FifFin

Proof. We denote by $\mathcal{A}$ the algebraic sum of all $S_{m,n}$. This is algebra which acts on the algebraic sum of all $\sum H(n)$. We know that $\mathcal{C}_c(G)(H)$ is a dense subspace of H which consists of differentiable vectors. The space $\mathcal{A}(H)$ is dense too. It follows that the projection to a $H(m)$ is dense. From Lemma 3.5 follows that this projection is contained in $\mathcal{A}(H)$. Hence it consists of differentiable vectors. □

3.7 Proposition. *Assume that $\pi : G \to \mathrm{GL}(H)$ is an irreducible representation on a Hilbert space. The algebra $S_{m,m}$ acts topologically irreducibly on $H(m)$ if this space is not zero.* SnnIrr

Proof. Let $h \in H(m)$ be a non-zero element. We want to show that $S_{m,m}h$ is dense in $H(m)$. (This means that $S_{m,m}$ acts topologically irreducible on $H(m)$.) We consider the space $\mathcal{A}h$. We know that $\mathcal{A}h$ is a dense subspace of H. It is contained in the algebraic sum $\sum H(n)$. We consider the projection of $\sum H(n)$ to $H(m)$. The image of $\mathcal{A}h$ under this projection is dense in $H(m)$. Lemma 3.5 shows that this image equals $S_{m,m}h$. This shows that $S_{m,m}$ acts topologically irreducible on $H(m)$. □

Since $S_{m,m}$ is abelian, we now obtain the following theorem.

3.8 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible representation on a Hilbert space. We assume that the restriction to K is unitary. Then H is the direct Hilbert sum of the spaces $H(n)$. Assume that $H(n)$ is finite dimensional. Then* $\dim H(n) \le 1$. *This is always the case if π is unitary.* TypI

We just mention that this a special case of a more general result that holds for any semi simple Lie group G and a maximal compact subgroup. Examples are $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n, \mathbb{R})$. For every irreducible unitary representation of G the K-isotypic components are finite dimensional. In other words: each irreducible unitary representation of K (which is always finite dimensional) occurs with finite multiplicity in $\pi|K$. The proof more involved, mainly since K is not commutative in general.

A vector $h \in H$ is called K-finite If the space generated by all $\pi(k)h$ is finite dimensional. The space of K-finite vectors is denoted by $H(K)$. The elements of $H_{m,n}$ are K-finite. Since every finite dimensional representation of a compact group is completely reducible, we obtain the following description.

3.9 Lemma. *Let π be a Banach representation of G on a Hilbert space H such that the restriction to K is unitary. Then* KeM

$$H(K) = \sum_{m \in \mathbb{Z}} H(m) \qquad \text{(algebraic sum).}$$

It is important to describe for a given irreducible unitary representation π the set of all n sich such that $H(n)$ is different from zero (and then one-dimensional). For this we look for operators that shift $H(n)$ which means that $H(n)$ is mapped into another $H(m)$. We will find such operators in the Lie algebra.

4. The Lie-algebra of the special linear group of degree two

We recall the exponential function for matrices $A \in \mathbb{C}^{(n,n)}$:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

It is clear that this series converges absolutely. The rule

$$\exp(A + B) = \exp(A) + \exp(B)$$

holds if A, B commute. The rule

$$B^{-1} \exp(A) B = \exp(B^{-1} A B)$$

is trivial. We need also the rule

$$\det \exp(A) = \exp(\operatorname{tr}(A))$$

which can be reduced to diagonal matrices (using the previous rule and the fact that the set of all matrices with n pairwise different eigenvalues is dense in the set of all matrices). We use the notation (Lie-bracket)

$$[A, B] = AB - BA.$$

Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup. We consider the set $\mathfrak{g}$ of all matrices A such that $\exp(tA) \in G$ for all $t \in \mathbb{R}$. It can be shown that $\mathfrak{g}$ is a Lie-algebra. This means that $\mathfrak{g}$ is a vector space and that $A, B \in \mathfrak{g}$ implies that $[A, B] \in \mathfrak{g}$. One can show that G is a smooth subset and that

$$\exp : \mathfrak{g} \longrightarrow G$$

is a local diffeomorphism at the origin. We do not need this general theory, since these facts can be verified directly for $G = \mathrm{SL}(2, \mathbb{R})$. From now on we restrict to this case.

In this case we have

$$\mathfrak{g} = \{A \in \mathbb{R}^{(2,2)}; \quad \operatorname{tr}(A) = 0\}.$$

Notice that it is trivial that this is a Lie-algebra. Besides $\mathfrak{g}$ we need also its complexification

$$\mathfrak{g}_{\mathbb{C}} := \{A \in \mathbb{C}^{(2,2)}; \quad \operatorname{tr}(A) = 0\}.$$

This is a complex Lie algebra (i.e. it is a complex vector space and invariant under the Lie-bracket).

We formulate a general fact.

4.1 Lemma. *The map* $\exp : \mathfrak{g} \to G$ *is locally topological at* 0*, i.e. it maps a suitable small open neighborhood of* $0 \in \mathfrak{g}$ *onto a small open neighborhood of the unit element in* G*.* ExpTop

In the case $\mathrm{SL}(2, \mathbb{R})$ the proof is very easy. One constructs an inverse of the exponential map by means of a matrix logarithm

$$-\log(E - A) = \sum_{n=1}^{\infty} \frac{A^n}{n}$$

which converges in a small neighborhood of $A = 0$. □

Finally we mention another result which is important in this connection.

4.2 Lemma. *The group $G = \mathrm{SL}(2,\mathbb{R})$ is generated by any neighborhood of the unit element.* GenNei

The following proof works for every connected group. (That $\mathrm{SL}(2,\mathbb{R})$ is connected follows from the Iwasawa decomposition). Let U be a an open neighborhood of the identity. By continuity the set of all a such that a and a^{-1} is contained in U is also an open neighborhood. Hence we can assume that $a \in U$ implies $a^{-1} \in U$. We consider the image $U(n)$ of

$$U^n \longrightarrow G, \quad (a_1,\dots,a_n) \longmapsto a_1 \cdots a_n.$$

The union G_0 of all $U(n)$ is an open subgroup of G. Since G is the disjoint union of cosets of G_0, the complement of G_0 in G is also open. Hence G_0 is open and closed in G and hence $G_0 = G$ since G is connected. □

5. The derived representation

Differential calculus usually is defined for maps $U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open subset. There is a straight forward generalization where $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ are replaced by Banach spaces, where in this context they are understood as Banach spaces over the field of real numbers. It is clear what this means. A map $f : U \to F$ in this context is called differentiable at $a \in U$ if there exists a continuous (real) linear map $L_a : E \to F$ such that

$$f(x) - f(a) = L_a(x-a) + r(x) \quad \text{where} \quad \lim_{x\to a} \frac{\|r(x)\|}{\|x-a\|} = 0.$$

If this is true for every $a \in U$ we call f differentiable. Then we can consider the derivative

$$Df : U \longrightarrow \mathrm{Hom}(E,F), \quad df(a) = L_a.$$

Since the subspace of bounded operators of $\mathrm{Hom}(E,F)$ is a Banach space too, we can ask for differentiability of df. In this way one can define the space of infinite differentiable functions $\mathcal{C}^\infty(U,F)$. As in the finite dimensional case, the chain rule holds for (infinitely often) differentiable functions. We also mention that a continuous linear map is differentiable by trivial reasons.

We want apply this to functions $G \to H$ where H is a Banach space (as usual over the complex numbers). Assume that $\pi : G \longrightarrow \mathrm{GL}(H)$ be a continuous representation. We associate to an arbitrary vector $h \in H$ a function

$$G \longrightarrow H, \quad x \longmapsto \pi(x)h.$$

We call the vector h differentiable if this function is infinitely often differentiable. We denote the space of differentiable vectors by H^∞. These is a sub-vector space. It depends of course on π. Hence, for example, H^∞_π is a more careful notation.

We give examples of a differentiable vector.

5.1 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and $f \in \mathcal{C}_c^\infty(G)$. Then the image of $\pi(f)$ is contained in H^∞. As a consequence the space H^∞ is a dense subspace of H.* ExDifV

Corollary. *Assume that H is a Hilbert space and that the restriction of π to K is unitary. Let m be an integer such that $\dim H(m) \le 1$. Then the elements of $H(m)$ are differentiable. (This applies if π is an irreducible unitary representation.)*

Proof. The first part follows from the formula

$$\pi(x)\pi(f)v = \int_G f(y)\pi(x)\pi(y)v dy = \int_G f(x^{-1}y)\pi(y)dy$$

by means of the Leibniz rule that allows to interchange integration and integration. (Of course we need a Banach valued version of the rule. We omit a proof of this, since it can be done as in the usual case.)

To prove the corollary we observe that $\pi(S_{m,m})H(m)$ is dense in $H(m)$ by Proposition 3.7. In the case that $H(m)$ is finite dimensional it is the whole of $H(m)$. Now we can apply the first part of the proof. □

Let $X \in \mathfrak{g}$ and $h \in H^\infty$. The map

$$\mathbb{R} \longrightarrow H, \quad t \longmapsto \pi(\exp(tX))h$$

is differentiable, since it is the composition of two differentiable maps. Hence we can define the operator $d\pi(X) : H^\infty \to H$:

$$d\pi(X)h := \frac{d}{dt}\pi(\exp(tX)h)\Big|_{t=0}.$$

This is related to a another construction, the Lie-derivative (from the left). This is for each $X \in \mathfrak{g}$ a map

$$\mathcal{L}_X : \mathcal{C}^\infty(G,H) \longrightarrow \mathcal{C}^\infty(G,H)$$

which is defined by

$$\mathcal{L}_X f(a) = \frac{d}{dt} f(a\exp(tX))\Big|_{t=0}.$$

(It is easy to show that $\mathcal{L}_X f$ is differentiable.) The Lie-derivative has nothing to do with the representation π. But we get a link to the derived representation if we apply it to functions of the type $x \mapsto \pi(x)h$.

5.2 Lemma. *Let $X \in \mathfrak{g}$ and $h \in H^\infty$. We consider the differentiable function $f(x) = \pi(x)h$ on G. The the formula* LieDer

$$\pi(a)d\pi(X)h = (\mathcal{L}_X f)(a)$$

holds, in particular

$$d\pi(X)h = (\mathcal{L}_X f)(e) \in H_\pi^\infty.$$

Proof. The second formula is just true by definition. The first one can be obtained if one applies $\pi(a)$ to the second one. One just has to observe that $\pi(a)$ commutes by continuity with the limit

$$\lim_{t\to 0} \frac{\pi(\exp(tX))h - h}{t}. \qquad \square$$

The Lie derivatives satisfy a basic commutation rule.

5.3 Proposition. *For $X, Y \in \mathfrak{g}$ the formula* DerKl

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X \qquad ([X,Y] = XY - YX)$$

holds.

Proof. The formula states

$$\frac{d}{dt} f(\exp(t[X,Y]))\big|_{t=0} =$$
$$\frac{d}{dt}\frac{d}{ds}\big(f(\exp(tX)\exp(sY)) - f(\exp(tY)\exp(sX))\big)\big|_{t=s=0}.$$

Here f is a $\mathcal{C}^\infty$ function on some open neighborhood of the unit element of $G = \mathrm{SL}(2,\mathbb{R})$. It is easy to show that f is the restriction of a $\mathcal{C}^\infty$-function on some open neighborhood of the unit element of $\mathrm{GL}(2,\mathbb{R})$ (which can be considered as an open subset of $\mathbb{R}^4$). Hence it is sufficient to prove the formula for $G = \mathrm{GL}(2,\mathbb{Z})$ and $\mathfrak{g}$ can be replaces we the space of all real 2×2-matrices. Using Taylor's formula one can reduce the proof to the case where f is a polynomial. The product rule shows that the formula is true for fg if it is true for f and g. Hence it is sufficient to prove it for linear functions. So we reduced the sstatement to the formula

$$\frac{d}{dt} \exp(t[X,Y])\big|_{t=0} =$$
$$\frac{d}{dt}\frac{d}{ds}\big(\exp(tX)\exp(sY)) - \exp(tY)\exp(sX)\big)\big|_{t=s=0}.$$

This is equivalent to the formula $[X,Y] = XY - YX$. $\square$

As a consequence of the commutation rule of the Lie derivative we obtain the following rule for the derived representation.

5.4 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. Then the following rule* DerCom

$$d\pi([X,Y]) = d\pi(X) \circ d\pi(Y) - d\pi(Y) \circ d\pi(X)$$

holds.

Propositions 5.3 and 5.4 provide special cases of the following definition.

5.5 Definition. *Let $\mathcal{A}$ be a an associative algebra (over the field of real numbers is enough). A map $\varphi : \mathfrak{g} \to \mathcal{A}$, $A \longmapsto \mathbf{A}$ is called a Lie homomorphism if it is $\mathbb{R}$-linear and if* LieHom

$$\varphi([A,B]) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$$

holds.

Hence Proposition 5.3 provides a Lie homomorphism

$$\mathfrak{g} \longmapsto \mathrm{End}(\mathcal{C}^\infty(G,H))$$

and Proposition 5.4 a Lie homomorphism

$$\mathfrak{g} \longmapsto \mathrm{End}(H^\infty).$$

In both cases the algebra on the right-hand side is a *complex* algebra (since H is a complex vector space and since we understand by End complex linear endomorphisms. In such a case we can extend φ to the complexification $\mathfrak{g}_\mathbb{C}$ by means of the formula

$$\mathfrak{g}_\mathbb{C} \longrightarrow \mathcal{A}, \quad \varphi(A) = \varphi(\mathrm{Re}(A)) + \mathrm{i}\varphi(\mathrm{Im}(A)).$$

It is easy to check that the formula

$$\varphi([A,B]) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$$

remains true where the bracket in $\mathfrak{g}_\mathbb{C}$ is of course defined by the formula $[A,B] = AB - BA$.

6. Explicit formulae for the Lie-derivatives

In the following we use the elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{g}$. Three of them give a basis of $\mathfrak{g}$. We also will consider the complexification $\mathfrak{g}_{\mathbb{C}}$. Here we use the (complex) basis

$$W, \quad E^- = H - \mathrm{i}V, \quad E^+ = H + \mathrm{i}V.$$

So we have

$$E^- = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}.$$

Recall that the Lie-derivatives to $\mathfrak{g}_{\mathbb{C}}$ can be extended by $\mathbb{C}$-linearity:

$$\mathcal{L}_{A+\mathrm{i}B} = \mathcal{L}_A + \mathrm{i}\mathcal{L}_B,$$

since H and hence $\mathcal{C}^\infty(G, H)$ is a complex vector space.

From the Iwasawa decomposition we know that we can write $g \in G$ in the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1}x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with unique x and $y > 0$. The angle θ is determined mod 2π. We need the expressions for x, y, θ in terms of a, b, c, d. To get them it is useful to use complex numbers. Let τ be a complex number in the upper half plane, $\operatorname{Im}\tau > 0$. Since c, d are real but not both zero, the number $c\tau + d$ is different from zero. Hence we can define

$$g(\tau) = \frac{a\tau + b}{c\tau + d}.$$

Let h be a second matrix from G. A direct computation which we omit shows

$$(gh)(\tau) = g(h(\tau)).$$

We also notice

$$g_\theta(\mathrm{i}) = \mathrm{i}.$$

Hence we obtain

$$\frac{a\mathrm{i} + b}{c\mathrm{i} + d} = x + \mathrm{i}y.$$

This gives us x and y in terms of a, b, c, d.

$$y = \frac{1}{c^2 + d^2}, \quad x = \frac{ac + bd}{c^2 + d^2}.$$

Looking at the second row of the Iwasawa decomposition we get

$$c\sqrt{y} = -\sin\theta, \quad d\sqrt{y} = \cos\theta.$$

This shows

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta = \frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

This gives as

$$\theta = \operatorname{Arg}\frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

Since θ is only determined mod 2π, we have to say a word about the choice of the argument Arg. All what we need is that for a given $g_0 \in G$ one can make the choice of Arg such it depends differentiably on g for all g in a small open neighborhood of g_0.

In the following we will fix $g \in G$ and $X \in \mathfrak{g}$ and consider

$$g(t) = g\exp(tX)$$

for small t. We write $x(t), y(t), \theta(t)$ in this case. As we mentioned the function $\theta(t)$ can be chosen for small t such that it depends differentially on t. If we insert $t = 0$ we get the original x, y, θ.

For the Lie-derivative we have to consider a differentiable function f on G. We can write it as function f of three variables. We get

$$f(g(t)) = F(x(t), y(t), \theta(t)).$$

By means of the chain rule we get

$$\frac{d}{dt}f(g(t)) = \frac{\partial F}{\partial x}\dot{x}(t) + \frac{\partial F}{\partial y}\dot{y}(t) + \frac{\partial F}{\partial \theta}\dot{\theta}(t).$$

Recall that we have to evaluate this expression at $t = 0$ to get the Lie derivative.

As an example we take

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$g(t) = \begin{pmatrix} a & b + ta \\ c & d + tc \end{pmatrix}.$$

We obtain

$$z(t) = \frac{a\mathrm{i} + at + b}{c\mathrm{i} + ct + d}.$$

Differentiation and evaluating at $t = 0$ gives

$$\dot{z}(0) = \frac{1}{(c\mathrm{i} + d)^2}.$$

Using the formulae

$$y = \frac{1}{c^2+d^2}, \qquad e^{2\mathrm{i}\theta} = \frac{(d-\mathrm{i}c)^2}{c^2+d^2}$$

we obtain

$$\dot z(0) = ye^{2\mathrm{i}\theta} \quad \text{or} \quad \dot x(0) = y\cos 2\theta,\ \dot y(0) = y\sin 2\theta.$$

Finally, to compute $\dot\theta(0)$, we use the formula

$$\cos\theta(t) = (d+ct)\sqrt{y(t)}.$$

Differentiation gives

$$-\dot\theta(t)\sin\theta(t) = (d+ct)\frac{\dot y(t)}{2\sqrt{y(t)}} + c\sqrt{y(t)}.$$

Evaluating by $t=0$ we get

$$\dot\theta(0)\sin\theta = \frac{d\dot y(0)}{2\sqrt{y}} - c\sqrt{y}.$$

We insert $-c\sqrt{y} = \sin\theta$ and $\dot y(0) = y\sin 2\theta = 2y\sin\theta\cos\theta$ to obtain

$$\dot\theta(0) = -d\sqrt{y}\cos\theta + 1 = -\cos^2\theta + 1 = \sin^2\theta.$$

Another – even easier example – is $\mathcal{L}_W$. A simple computation gives

$$\exp tW = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence we obtain that $\mathcal{L}_W$ is given by the operator $\partial/\partial\theta$. In a similar way other elements of the Lie algebra can be treated. Since $X = 2X - W$ we get now $\mathcal{L}_V$. We omit the computation for H and just collect the formulae together.

6.1 Proposition. *Let $f \in \mathcal{C}^\infty$ and $A \in \mathfrak{g}$. We denote by $F(x,y,\theta)$ the corresponding function in the coordinates and similarly $G(x,y,\theta)$ for $g = \mathcal{L}_A f$. The operator $F \mapsto G$ can be described explicitly as follows:* LieDE

$$\begin{aligned}
\mathcal{L}_X &= y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + \sin^2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_W &= \frac{\partial}{\partial\theta},\\
\mathcal{L}_V &= 2y\cos 2\theta\frac{\partial}{\partial x} + 2y\sin 2\theta\frac{\partial}{\partial y} - \cos 2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_H &= -2y\sin 2\theta\frac{\partial}{\partial x} + 2y\cos 2\theta\frac{\partial}{\partial y} + \sin 2\theta\frac{\partial}{\partial\theta},
\end{aligned}$$

and, as a consequence,

$$\mathcal{L}_{E^-} = -2\mathrm{i}ye^{-2\mathrm{i}\theta}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big) + \mathrm{i}e^{-2\mathrm{i}\theta}\frac{\partial}{\partial\theta}.$$

7. Analytic vectors

Let E, F be Banach spaces over the field of real numbers and let $U \subset E$ be an open subset. We introduced the notion of a differentiable map $U \to F$ In the case that E is finite dimensional (but F may be not) we can also define the notion of an analytic map. In the case $E = \mathbb{R}^n$ this means as usual that for each $a \in U$ there exists a small neighborhood in which there exists an absolutely convergent expansion as power series

$$f(x) = \sum_{\nu \in \mathbb{N}_0^n} a_\nu (x_1 - a_1)^{\nu_1} \cdots (x_n - a_n)^{\nu_n} \qquad (a_\nu \in F).$$

This notion is invariant under linear transformation of the coordinates, hence it carries over to arbitrary E. We denote by $\mathcal{C}^\omega(U, F)$ the space of all analytic functions. This is a subspace of $\mathcal{C}^\infty(U, F)$. The basic property of analytic functions is the principle of analytic continuation. Assume that U is connected and that $a \in U$ a point that all derivatives of f or arbitrary order vanish (this is understood to include $f(a) = 0$). Then f is identically zero.

Using the standard coordinates of G, we can define the notion of analytic function $G \to H$ into any Banach space. If $\pi : G \to \mathrm{GL}(H)$ is a representation we can define the notion of an analytic vector $h \in H$. By definition this means that the function $\pi(x)h$ on G is analytic. The set H^ω of all analytic vectors is a sub-vector space of H^∞.

We recall the formula for the Lie-derivative

$$(\mathcal{L}_X f)(y) = \frac{d}{dt} f(y \exp(tX))\Big|_{t=0}.$$

We replace y by $y \exp(uX)$ and obtain

$$(\mathcal{L}_X f)(y \exp(uX)) = \frac{d}{dt} f(y \exp((u+t)X))\Big|_{t=0} = \frac{d}{du} f(y \exp(uX)).$$

By induction follows

$$(\mathcal{L}_X^n f)(y \exp(uX)) = \frac{d^n}{du^n} f(y \exp(uX)).$$

The Taylor expansion of the function $t \mapsto f(y \exp(tX))$ is given by

$$\begin{aligned} f(y \exp(tX)) &= \sum_{n=0}^\infty \frac{1}{n!} \frac{d^n}{dt^n} f(y \exp(tX))\Big|_{t=0} t^n \\ &= \sum_{n=0}^\infty \frac{1}{n!} (\mathcal{L}_X^n f)(y)\, t^n. \end{aligned}$$

This formula is true for given X, y if $|t|$ is sufficiently small, $|t| < \varepsilon$. For a real constant the formula $\mathcal{L}_{cX} = c\mathcal{L}_X$ can be checked. This shows that (for fixed y) the Taylor formula holds if X is in a sufficiently small neighborhood of the origin. We specialize the Taylor expansion to the function $f(x) = \pi(x)h$ and to $t = 1$.

7.1 Proposition. *Let $h \in H$ be an analytic vector. For sufficiently small X the formula* DerExp

$$\pi(\exp(X))h = \sum_{n=0}^{\infty} \frac{1}{n!} d\pi(X)^n h$$

holds.

Making use of Lemma 4.1 and Lemma 4.2 we now obtain the following important result.

7.2 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and let $V \subset H$ be a linear subspace consisting of analytic vectors that is invariant under $d\pi(\mathfrak{g})$. Then the closure of V is invariant under G.* LieAnIrr

In the next section we will prove the existence of analytic vectors.

8. The Casimir operator

In the following we will make use of the basic commutation rules in $\mathfrak{g}$:

$$[E^+, E^-] = -4\mathrm{i}W, \quad [W, E^+] = 2\mathrm{i}E^+, \quad [W, E^-] = -2\mathrm{i}E^-.$$

The can be verified by direct computation.

Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and

$$\varrho : \mathfrak{g} \longrightarrow \mathcal{A}$$

be a Lie homomorphism, i.e. a linear map with the property

$$\varrho([A, B]) = \varrho(A)\varrho(B) - \varrho(B)\varrho(A).$$

We also can consider its $\mathbb{C}$-linear extension $\mathfrak{g}_{\mathbb{C}} \to \mathcal{A}$. Our typical example is that $\mathcal{A}$ is the algebra of (algebraic) endomorphisms of an abstract (complex) vector space $\mathcal{H}$. In this case we talk about a Lie-algebra representation of $\mathfrak{g}$ on $\mathcal{H}$. We denote the image of element $A \in \mathfrak{g}_{\mathbb{C}}$ by the corresponding bold letter $\boldsymbol{A}$. We define the *Casimir element* by

$$\omega = \boldsymbol{H}^2 + \boldsymbol{V}^2 - \boldsymbol{W}^2.$$

Using the above commutation rules we can check by a simple computation

$$\omega = \boldsymbol{H}^2 + \boldsymbol{V}^2 - \boldsymbol{W}^2 = \boldsymbol{E}^+\boldsymbol{E}^- + 2\mathrm{i}\,\boldsymbol{W} - \boldsymbol{W}^2.$$

The basic property of the Casimir element is that it commutes with the image of $\mathfrak{g}_{\mathbb{C}}$.

8.1 Lemma. *The Casimir element ω (with respect to $\varrho : \mathfrak{g} \to \mathcal{A}$) commutes with all $\boldsymbol{A}$ for $A \in \mathfrak{g}$.* CasCom

Proof. On uses the second formula for the Casimir operator and applies the above commutation rules. □

The Lie algebra $\mathfrak{g}$ acts in the space $\mathcal{C}^\infty(G, H)$. Hence we can consider the Casimir operator ω acting on this space. Using the formulae on Proposition 5.2 we get the explicit expression

$$\omega = 4y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 4y\frac{\partial^2}{\partial x \partial \theta}.$$

We are especially interested on its action of functions of the type

$$f(xk) = e^{\mathrm{i}n\theta} f(x).$$

Since $\partial f/\partial\theta = \mathrm{i}nf$ we see that the operator

$$4y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 4y\mathrm{i}n\frac{\partial}{\partial x}$$

has the same effect on f as the Casimir operator. The advantage of the latter operator is that it is – like the Laplace operator – an *elliptic differential operator.* We make use of a basis result that $\mathcal{C}^\infty$-eigen functions $Df = \lambda f$ of an elliptic differential operator D are analytic functions. Usually this theorem is formulated scalar valued function. But it is also true for Banach valued functions. For this one can use for example the following general result.

8.2 Proposition. *A function $f : G \to H$ is analytic if an only if $L \circ f$ is analytic for every continuous linear function L.* WeaAn

We do not give a proof. □

Let now $\pi : G \to \mathrm{End}(H)$ a Banach representation and let $h \in H$ by a differentiable vector. Then there is the Casimir operator acting on H^∞. Assume that h is an eigen vector and that $h \in H(m)$. We claim that h is an analytic vector. We have to show that the function $f_h(x) = \pi(x)h$ is analytic. The condition $h \in H(m)$ implies

$$f_h(xk) = e^{\mathrm{i}n\theta} f_h(x).$$

For any $A \in \mathfrak{g}$ we have

$$\mathcal{L}_A f_h = f_{d\pi(A)h}.$$

This carries over to the Casimir operator. So we can write

$$\omega f_h = f_{\omega h}.$$

By assumption h is an eigen vector of the Casimir operator. This implies that f_h is an eigen function. So we get that f_h is analytic. By definition this means that h is analytic. This gives the following result.

8.3 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and let $h \in H(m)$ be a differentiable vector which is an eigen vector of the Casimir operator. Then h is analytic.* CasAn

This gives us the possibility to identify many analytic vectors. For this we have to study the action of the generators of $\mathfrak{g}$ on the spaces $H(m)$ in more detail.

8.4 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation on a Hilbert space H. We assume that the restriction to K is unitary. We also assume that the elements of $H(m)$ are differentiable. Then $d\pi(W)$ acts on $H(m)$ by multiplication by* $\mathrm{i}m$. *The operator $d\pi(E^+)$ maps $H(m)$ to $H(m+2)$ and $d\pi(E^-)$ maps $H(m)$ to $H(m-2)$.* LemErz

Proof. Recall that the space H is the direct Hilbert sum of the K-isotypic components, which are $H(m)$. A direct computation gives

$$\exp(tW) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

This gives

$$\pi(\exp(tW))h = e^{\mathrm{i}mt}h \quad \text{for} \quad h \in H(m).$$

If we differentiate by t and evaluate than by $t = 0$ we get the desired result for the action of W.

To get the statement for the action of E^+ we use the rule

$$d\pi(W)d\pi(E^+) = d\pi[W, E^+] + d\pi(E^+)d\pi(W)$$

and the commutation rule $[W, E^+] = 2\mathrm{i}E^+$. For a vector $a \in H(m)$ we get

$$d\pi(W)d\pi(E^+)h = \mathrm{i}(m + 2)d\pi(E^+)h.$$

Hence $d\pi(E^+)h$ is an eigen value of $d\pi(W)$ with eigenvalue $\mathrm{i}(m + 2)$. Hence it must lie in $H(m + 2)$. The argument for E^- is similar. □

From the second formula for the Casimir we see that $H(m)$ is mapped into itself. This gives the following basic result.

8.5 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$ be an unitary representation such that all $H(m)$ have dimension ≤ 1. (This is the case if π is irreducible). Then the vectors from $H(K) = \sum H(m)$ (algebraic sum) are analytic and this space is invariant under $\mathfrak{g}$.* HKinv

Proof. Since the spaces $H(m)$ have dimension ≤ 1 they consist of differentiable vectors (Lemma 3.6). The elements of $H(m)$ are eigen elements of the Casimir operator. □

By a representation of the Lie algebra $\mathfrak{g}$ on the abstract vector space E we understand a Lie homomorphism map $\pi : \mathfrak{g} \to \operatorname{End}(E)$, i.e. a linear map with the property

$$\pi([X,Y]) = \pi(X) \circ \pi(X) - \pi(Y) \circ \pi(X).$$

For a unitary representation $\pi : G \to \mathrm{GL}(H)$ we can consider the derived representation

$$d\pi : \mathfrak{g} \longrightarrow \operatorname{End}(H(K)).$$

8.6 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$, be a unitary representation such that all $H(m)$ have dimension ≤ 1. Then π is irreducible if and only if the derived representation* DerIrr

$$d\pi : \mathfrak{g} \longrightarrow \operatorname{End}(H(K))$$

has the following property. Let $\mathcal{A}$ be the algebra of operators that is generated by the image of $\mathfrak{g}$ and by the identity. For each non-zero h which is contained in some $H(m)$ we have $\mathcal{A}(h) = H(K)$.

Proof. We notice the the set $\mathcal{A}(h) = \{A(h), \quad A \in \mathcal{A}\}$ is a vector space. This vector space can also be described as follows. Let $X = X_1 \cdots X_n$ be an operator such that each X_i is one of the $d\pi(E^+)$, $d\pi(E^-)$, $d(W)$. Then $\mathcal{A}(h)$ is the vector space generated by all $X(h)$.

For a X as above the space $X(\mathbb{C}h)$ is either 0 or it is one the $H(n)$. Hence we see that $\mathcal{A}(h)$ is generated by certain spaces $H(n)$.

Now we assume that π is irreducible. We prove that $\mathcal{A}(h)$ is the full $H(K)$. We argue indirectly. So we can assume that there exist an $H(n) \ne 0$ which is not contained in $\mathcal{A}(h)$. We recall that the spaces $H(k)$ are pairwise orthogonal. Hence the preceding remark shows that $H(n)$ is orthogonal to $\mathcal{A}(h)$. But then $H(n)$ is orthogonal to the closure of $\mathcal{A}(h)$. We know that this space is invariant under G. But this is not possible since we assumed that π is not the trivial one dimensional representation.

Assume now that $\mathcal{A}(h) = H(K)$ for all nonzero $h \in H(m)$. We claim that π is irreducible. Again we argue indirectly. We find a proper closed invariant subspace H'. We can take a non-zero isotypic component $H'(m)$. Consider a non-zero element $h \in H'(m)$. By assumption then $\mathcal{A}(h)$ is $H(K)$. This shows $H(K) \subset H'$ and hence $H = H'$. □

Since the Casimir operator commutes with all elements of $\mathfrak{g}$ we obtain the following kind of a Schur lemma.

8.7 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a irreducible unitary representation. Then the Casimir operator acts on $H(K)$ by multiplication by some constant.* CasEig

This constant is a basic invariant of π.

9. Admissible representations

Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation on a Hilbert space such that the restriction to K is unitary. As we know, $d\pi(W)$ acts on $H(n)$ by multiplication with $\mathrm{i}n$. Recall that $\dim H(n) \le 1$ if π is an irreducible unitary representation. We are led to consider the following algebraic objects.

9.1 Definition. *An (algebraic) representation of $\mathfrak{g}$ on a on a vector space $\mathcal{H}$ is called admissible if it is not the zero representation and if the following conditions are satisfied.* AdmRep

1) *The eigen values of W are contained in $\mathrm{i}\mathbb{Z}$ and the eigen spaces $H(n)$ of the operator W with eigen value $\mathrm{i}n$, $n \in \mathbb{Z}$ are of dimension ≤ 1.*
2) *Let $\mathcal{A}$ be the algebra of operators that is generated by the image of $\mathfrak{g}$. For each non-zero h which is contained in some $H(m)$ we have $\mathcal{A}(h) = \mathcal{H}$.*
3) *The space $\mathcal{H}$ is the (algebraic) direct sum of the subspaces $H(n)$.*

It is clear what an isomorphism of admissible representation means. We emphasize that this is understood in a pure algebraic way. As we have seen, every irreducible unitary representation $G \to \mathrm{GL}(H)$ has an underlying admissible representation of $\mathfrak{g}$ on $\mathcal{H} = H(K)$. It is clear that unitary isomorphic representations have underlying isomorphic admissible representation. We will see that the converse is also true.

Before we prove this, we formulate a simple lemma.

9.2 Lemma. *For a unitary representation $\pi : G \to \mathrm{GL}(H)$ the operators $d\pi(X)$, $X \in \mathfrak{g}$ are skew hermitian.* SkHer

We recall that $A : H \to H$ is called skew hermitian if $\langle Aa, b\rangle = -\langle a, Ab\rangle$. Lemma 9.2 is an immediate consequence of the definition of the derived representation. □

9.3 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$, $\pi' : G \to \mathrm{GL}(H')$ be two irreducible representations such that the underlying admissible representations are (algebraically) isomorphic. Then π, π' are isomorphic as unitary representations.* InfIs

Proof. Let $T : H(K) \to H'(K)$ be an isomorphism of the admissible representations. We choose a non zero $h \in H$ which is contained in some $H(m)$. We normalize h such that $\langle h, h\rangle$. We set $h' = Th$. Without loss of generality we may assume that $\langle h', h'\rangle = 1$ since we can replace the scalar product of H' by a multiple. Now we claim that T preserves the scalar products. For the proof we use the property 2) in Definition 9.1. It implies that $H(K)$ is generated by all $A_1 \cdots A_n h$, where $A_i \in \mathfrak{g}$. So we we have to show that T preserves the scalar products for such elements. This is done by induction. We just explain the beginning to the induction to give the idea. Since $H(m)$ is one-dimensional and since the spaces $H(n)$ are pairwise orthogonal, we know all scalar products

$\langle h, x\rangle$. Let $A \in \mathfrak{g}$. The formula $\langle Ah, x\rangle = -\langle h, Ax\rangle$ gives all scalar products $\langle Ah, x\rangle$. Proceeding in this way we get that all scalar products are determined (from $\langle h, h\rangle = 1$). The same calculation can be done in H'. In this way we can see that T preserves the scalar products. Now we can extend to an isomorphism of Hilbert spaces $T : H \to H'$. From Proposition 7.1 in connection with the Lemmas 4.2 and 4.1 we obtain that T preserves the action of G. □

The classification of the irreducible unitary representations of G is divided into three steps.

1) Classify the admissible representations of $\mathfrak{g}$.
2) Exhibit those for which the underlying space $\mathcal{H}$ admits a Hermitian scalar product such that the elements of $\mathfrak{g}$ acts as skew hermitian operators.
3) Prove that they all can be realized by an irrreducible unitary representation of G.

We study in detail admissible representations. For this we will use the basis E^+, E^+, W for $\mathfrak{g}_{\mathbb{C}}$. We recall

$$[E^+, E^-] = -4\mathrm{i}W, \quad [W, E^+] = 2\mathrm{i}E^+, \quad [W, E^-] = -2\mathrm{i}E^-.$$

We recall also (Lemma 8.4)

$$E^+(H(n)) \subset H(n+2), \quad E^-(H(n)) \subset H(n-2).$$

We see that the spaces

$$H^{\text{even}} = \sum_{n \text{ even}} H(n), \quad H^{\text{odd}} = \sum_{n \text{ odd}} H(n)$$

are invariant subspaces. Hence we have to distinguish between and even case (all $H(2n+1)$ are zero) and an odd case (all $H(2n)$ are zero).

Let S be a set of all integers which are all odd or all zero. We call S an interval if for $m, n \in S$ each number of the same parity between m and n is contained in S. We claim now that the set S of all n such that $H(n) \neq 0$ is an interval. To prove this we consider an $n \in S$ such that $H(n)$ is different from zero. Recall that $H(n)$ is one-dimensional. We choose a generator h. The space H is generated by all $A_1 \dots A_m h$ where $A_i \in \mathfrak{g}_{\mathbb{C}}$. From the relations between the generators we see that H is generated by $E_+^m h$ and $E_-^m h$. Let for example $H(n+2k) = 0$, $k > 0$. Then $E_+^{n+2k} h = 0$ and hence all $H(m)$, $m > n+2k$, are zero. Hence S is an interval.

9.4 Proposition. *For the set S of integers m with the property $H(m) \neq 0$ of an infinite dimensional admissible representation there are the following possibilities:* AdmS

1) *S is the set of all even integers.*
2) *S is the set of all odd integers.*
3) *There exists $m \in S$ such that S consists of all $x \geq m$ with the same parity.*
4) *There exists $n \in S$ such that S consists of all $x \leq n$ with the same parity.*
5) *There exist integers $m \leq n$ of the same parity such that S consists of all $x \in \mathbb{Z}$, $m \leq x \leq n$, of the same parity.*

In the cases 3)-5) we call m the lowest weight and the non-zero elements of $H(m)$ the lowest weight vectors. Similarly we call n the highest weight.

We study case 1) in more detail. We choose a non-zero vector $h \in H(0)$. We know that E^+ is non zero on all $H(n)$ (n even). Hence we can define for all even n a uniquely determined $h_n \in H(n)$ such that

$$h_0 = h, \quad E^+ h_n = h_{n+2}.$$

Then we define the number $c_n \neq 0$ by

$$E^- h_n = c_{n-2} h_{n-2}.$$

The system of numbers $(c_n)_{n \text{ even}}$ is independent of the choice of h. It is clear that the action of $\mathfrak{g}_{\mathbb{C}}$ is determined by this system of numbers and it is also clear that isomorphic representations lead to the same system. A much better result is true. The relation $[E^+, E^-] = -4\mathrm{i}W$ shows

$$c_n - c_{n+2} = 4n.$$

Hence all c_n are determined by one, for example by c_0. We see that the isomorphy type of the representation is determined by one parameter! The case 2) is similar. We obtain the following result.

9.5 Proposition. *Consider an admissible representation of type 1). Then there exists a parameter $c \neq 0$ such that* ParTyp

$$E^+ E^- h = ch \quad \textit{for} \quad h \in H(0)$$

The representation is determined up to isomorphism by c. In the case of type 2) the same result holds if one replaces $H(0)$ by $H(1)$.

Assume now that there is a highest weight n. Again we choose a non-zero $h \in H(n)$. As in the previous case we can consider the parameter c defined by $E^+E^-h = ch$. But now $E^+h = 0$. Hence the relation $[E^+, E^-] = -4\mathrm{i}W$ gives

$$c = 4m.$$

9.6 Proposition. *An admissible representation with a highest weight vector is determined up to isomorphy by its highest weight m. If h is a highest weight vector then* LowUniq

$$E^+ E^- h = 4mh.$$

The same is true for lowest weight representations if one replaces the equation by

$$E^- E^+ h = 4nh.$$

We mention that in the case 5) there exists a highest and a lowest weight.

10. The Bargmann classification

Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representations. Then the derived representation $\mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(H(K))$ has the property that the operators $d\pi(A)$ for $A \in \mathfrak{g}$ are skew symmetric. Hence it is natural to ask for an admissible representation $\mathfrak{g}$ on $\mathcal{H}$ whether there exists a hermitian scalar product on $\mathcal{H}$ such that the elements of $\mathfrak{g}$ act skew-symmetrically. If this is case we call (by some misuse of language) the representation unitarizible.

10.1 Proposition. *Consider a unitarizible admissible representation. If it has no highest or lowest weight then its parameter c (s. Proposition 9.5) is real and negative.* WeiUni

If it has a lowest weight n then $n > 0$. If it has a highest weight n then $n < 0$.

Corollary. *If there exists a lowest or highest weight then $H(0)$ is 0. So $H(0)$ is different form zero only in the case 1) of Proposition 9.4.*

Proof. First we treat the case where there exists no highest or lower weight vector. We restrict to the even case. (The odd case is similar.) Hence we have a non zero element $h \in H(0)$ and the consntant c is defined by $E^+E^-h = ch$. From the condition that the real elements $A \in \mathfrak{g}$ act skew hermitian we obtain the rule $\langle E^+h, h'\rangle = -\langle f, E^-h'\rangle$ and hence

$$\langle E^-h, E^-h\rangle = -\langle h, -E^+E^-h\rangle = -\langle h, ch\rangle = -\bar{c}\langle h, h\rangle.$$

It follows that c is real and negative.

A similar argument works in the case of a highest weight m. Let $h \in H(m)$ a non-zero element. Then

$$\langle E^-h, E^-h\rangle = -\langle h, E^+E^-h\rangle = -m\langle h, h\rangle.$$

This shows that m is negative. (The alternative $m = 0$ is not possible since that we would have $H = H(m)$ and $\mathfrak{g}$ would act identically zero.) □

We also see that for unitarizible admissible representation there can not exist a highest and a lowest weight.

This gives the classification of unitarizible admissible representations.

10.2 Theorem. *For each negative real number number $c < 0$ there exists an even and an odd unitarizible admissible representation without highest or lowest weight with this parameter. Each integer $n < 0$ occurs as highest weight and and each integer $n > 0$ occurs as lowest weight of a unitarizable admissible representation. Every non-trivial unitarizible admissible representation is isomorphic to exactly one representation of this list.* RealAdm

Proof. It remains to prove the existence of these representations. We take for example the case of an even representation without highest or lowest weight. We take for each even n a one-dimensional vector space $H(n)$ and define $\mathcal{H}$ to be the algebraic sum of $H(n)$. Then we define the operators W, E^+, E^- by the necessary formulae which we derived above. It is rather clear that this gives an admissible representation. The definition of the scalar product is also clear. The other cases are similar. □

The question arises whether each admissible representation can be realized by unitary representations of G (in the sense that it is isomorphic to its derived representation).

10.3 Theorem. *Each unitarizible admissible representation can be realized by an irreducible unitary representation.* AdmReal

The proof will be given in the next section by means of explicit constructions in each of the cases. □

We recall that an irreducible unitary representation is determined up to unitary equivalence by the isomorphy type of the underlying admissible representation.

Collecting together, we get the classification of the irreducible unitary representations of G. First we introduce the usual notations. Recall that in case 1) and case 2) the representation is determined by a single parameter $c < 0$ which can be arbitrary in both cases. Instead of c we will use a new parameter s.

In case 1) (even case, no highest or lowest weight) we define s by

$$(s+1)(s-1) = c.$$

The number $(s+1)(s-1)$ is a negative real number if and only if

1) s is purely imaginary ($c \leq -1$)
2) s is real and $-1 < s < 1$. ($-1 < c < 0$)

We notice that the solution s is unique up to the replacement $s \mapsto -s$. We use the following notation:

The even principal series consists of all representations of even type without highest or lowest weight and with the property that s is purely imaginary.

The complementary series consists of all representations of even type without highest or lowest weight and with the property that $s \in (-1, 1)$ but $s \neq 0$.

Next we treat case 2) (odd case, no highest or lowest weight). In this case we introduce s by

$$s^2 = c.$$

Hence s is different from zero and purely imaginary. It is determined up to sign.

The odd principal series consists of all representations of odd type without highest or lowest weight and with the property that s is different from zero and purely imaginary.

In the case 3) there exists a lowest weight $m \le -1$ and in the case 4) there exists a highest weight $n \le -1$.

The border cases with highest weight -1 or lowest weight 1 have some special properties. Hence they are separated from the other representations with a highest or lowest weight vector. Those with $m \le -2$ or $n \ge 2$ define the *discrete series* and the two with $m = -1$ or $n = 1$ define the mock discrete series.

Collecting together we obtain Bargmann's classification of all irreducible representations π of G. If this representation is not the trivial one dimensional representation then $\mathfrak{g}$ acts non identically zero (Proposition 2.7.1) and then the derived representation is a unitarizible admissible representation. The above discussion gives now the main result.

10.4 Theorem. *Each unitary irreducible unitary representation of $G = \mathrm{SL}(2, \mathbb{R})$ is either the trivial one-dimensional representation or it is unitary isomorphic to a representation of the following list.* BargM

1) *The even principal series, $s \in \mathrm{i}\mathbb{R}$,*
2) *the odd principal series, $s \in \mathrm{i}\mathbb{R} - \{0\}$,*
3) *the complementary series, $s \in (-1, 1) - \{0\}$,*
4) *the discrete series with highest weight $m \le -2$ or lowest weight $n \ge 2$.*
5) *the mock discrete series (two representations, (highest weight -1 or lowest weight 1).*

In the first three cases, s is determined up to its sign. In the last two cases the weight is uniquely determined.

Why has the mock discrete series been separated from the discrete series? If G is an arbitrary locally compact group, one has a general notion of a discrete series representation. An irreducible unitary representation is called a discrete series representation of it occurs (as unitary representation) in the regular representation $L^2(G)$. It can be shown that the discrete series representations of $G = \mathrm{SL}(2, \mathbb{R})$ in this sense consist of all representations with a higher or lower weight vector with two exceptions, the weights 1 and -1 do not occur. Hence these play a special role. Since they look similar as the discrete series representations they are called "mock discrete".

We mentioned already that for an irreducible unitary representation the Casimir operator acts on $H(K)$ by multiplication with some constant λ.

10.5 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representation. The Casimir operator acts in $H(K)$ by multiplication with a constant λ. We have $\lambda = (s + 1)(s - 1)$ for both principal and for the complementary series.* CasLa

The proof is given by a trivial calculation.

11. The principal and the mock discrete series

We consider the representations $H(s)$ which have been described in Chap. I, Sect. 7.

11.1 Lemma. *Let $f \in \mathcal{H}(s)$ be an element that is $\mathcal{C}^\infty$ considered as function on the group G. Then the image of f in $H(s)$ is a $\mathcal{C}^\infty$-vector of the representation π_s and we have* DiffFL

$$\mathcal{L}_X f = d\pi_s(X) f.$$

Proof. It is easy to check that $\mathcal{L}_X f$ has the transformation properties of functions from $\mathcal{H}(s)$. Since it is continuous it is contained in $\mathcal{H}(s)$. We have to show that

$$\lim_{t\to 0} \int_K \Big| \frac{f(k\exp(tX)) - f(k)}{t} - \mathcal{L}_X f(k)\Big|^2 dk = 0.$$

by definition of $\mathcal{L}$ the integrand tends pointwise to 0. Using the mean-value theorem it is easy to show that the integrand is bounded for small t. Hence the Lebesgue limit theorem can be applied. □

The space $H(n,s)$ of all K-eigenfunctions which pick up the nth power of the standard character is one dimensional and generated by the function

$$\varphi\left(\begin{pmatrix} \sqrt{y} & * \\ 0 & \sqrt{y}^{-1}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right) = y^{(s+1)/2} e^{in\theta}.$$

We can use the formula in Proposition 6.1 to compute the derived representation. The result is

$$\boxed{\begin{aligned} d\pi_s(W)\varphi_n &= in\varphi_n, \\ d\pi_s(E^-)\varphi_n &= (s+1-n)\varphi_{n-2} \\ d\pi_s(E^+)\varphi_n &= (s+1+n)\varphi_{n+2} \end{aligned}}$$

From this description, it is easy to to invariant subspaces, namely

$$H(s)^{\text{even}} = \widehat{\bigoplus_{n \text{ even}}} \mathbb{C}\varphi_n, \quad H(s)^{\text{odd}} = \widehat{\bigoplus_{n \text{ odd}}} \mathbb{C}\varphi_n.$$

We first treat the even case. Then the parameter c from the previous section computes as $c = (s+1)(s-1)$. This was the reason that we introduced already somewhat the parameter s as solution of this equation. We recall that the representation π_s is unitary if $\operatorname{Re} s = 0$. We see that the corresponding derived representation is the even principal series. But one cannot realize the complementary series in this way, since this would demand $s \in (-1, 1)$, $s \neq 0$. But in this case π_s is only a Banach representation. Hence one needs for the complementary series a different kind of realization. We indicated it in Chapt. I, Sect. 7. We will give more details.

The odd principal series is obtained completely from π_s. Here the parameter c is computed as $c = s^2$ which again explains the conventions from the previous section. Hence in the case that s is purely imaginary but $s \neq 0$ In this way we get realizations of the two principal series where $s = 0$ has been excluded.

In the case $s = 0$ the odd space can be decomposed into subspaces again. We obtain in the case $s = 0$ two irreducible subspaces of the odd space,

$$\widehat{\bigoplus_{n\geq 1 \text{ odd}}} \mathbb{C}\varphi_n, \qquad \widehat{\bigoplus_{n\leq 1 \text{ odd}}} \mathbb{C}\varphi_n.$$

Obviously they are realizations of the two mock discrete representations. So in this sense the mock discrete representations are simply degenerations of the principal series.

Hence we have found realizations of the principal series and the tow mock discrete representations and we mention that the complementary series can also be realized by concrete unitary representations.

It remains to realize the discrete series. The holomorphic discrete series gives the discrete series with a lowest weight and the antiholomorphic discrete series gives the discrete series with a highest weight.

12. Automorphic forms

We consider a discrete subgroup $\Gamma \subset G$ with compact quotient $\Gamma\backslash G$. We recall that the representation of G on $L^2(\Gamma\backslash G)$ is completely reducible with finite multiplicities. One can ask which representations of the Bargmann list occur and for their multiplicity. For this we make a simple remark.

12.1 Lemma. *Let $f : \mathbb{H} \to \mathbb{C}$ be a function on the upper half plane and let m be an integer. We consider the function* HochHeb

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\Big(\frac{a\mathrm{i} + b}{c\mathrm{i} + d}\Big)(c\mathrm{i} + d)^{-m}.$$

then we have

$$F(gk_\theta) = e^{im\theta}F(g)$$

and every function on G with this transformation property comes from a function f on $\mathbb{H}$. *Moreover we can write F as*

$$F\left(\begin{pmatrix}\sqrt{y} & \sqrt{y}^{-1}x \\ 0 & \sqrt{y}^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right) = f(x+\mathrm{i}y)e^{\mathrm{i}m\theta}.$$

The function F is right-invariant under Γ *if and only if f satisfies*

$$f(az+b)(cz+d)^{-1} = (cz+d)^m f(z)$$

for all elements in Γ.

Proof. The proof is straight forward. □

Now we assume that their is an irreducible closed subspace $H \subset L^2(\Gamma\backslash G)$ which belongs to the holomorphic discrete series with lowest weight $m \geq 2$. Wir consider a non-zero lowest weight vector h. From Lemma 12.1 we know that h comes from a function $f : \mathbb{H} \to \mathbb{C}$ with the transformation property

$$f(az+b)(cz+d)^{-1} = (cz+d)^m f(z), \qquad \begin{pmatrix}a & b \\ c & d\end{pmatrix} \in \Gamma.$$

Since h is a lowest weight vector we have $E^- h = 0$. Using the explicit formula for E^- we obtain that

$$\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big) f = 0.$$

This means that f is holomorphic. This means that f is a holomorphic automorphic form. Conversely it can be shown that every holomorphic automorphic form occurs in this way.

But als other representations of the Bargmann list may occur. For example assume that an even principal series representation with parameter s occurs. We can now consider a non zero vector h of weight 0. This is invariant under K and corresponds to a function f on the upper half plane. Recall that h is an eigen form of the Casimir operator with eigen value $\lambda = (s+1)(s-1)$. Looking at the explicit expressions for $E^{\pm}$ we see that this means

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f = \lambda f.$$

What we have found is so-called wave form in the sense of Maass. Mass gave a generalization of the theory of modular forms replacing holomorphicity by certain differential equation. All these Maass forms can be recovered in the following way: Consider an irreducible sub representation $H \subset L^2(\Gamma\backslash G)$. Take a vector $h \in H(m)$ for an arbitrary m. By Lemma 12.1 this corresponds to function f on $\mathbb{H}$ with a certain transformation property. Make use of the fact that h is an eigen form of the Casimir operator. This produces a differential equation for f. In this way on recovers precisely the differential equations that Maass has introduced.

Chapter III. Representations of the Poincaré group

1. Unitary representations of some compact groups.

In this section we mention some results about the representation theory of the groups U(n) and SU(n). Here U(n) denotes the group of all $n \times n$-matrices A with the property $\bar{A}'A = E$ and SU(n) denotes the subgroup of elements of determinant one. We want to describe the unitary irreducible representations of them.

We recall some facts for compact groups K:

1) Each irreducible unitary representation of a compact group is finite dimensional.
2) If $K \to \mathrm{GL}(V)$ is a finite dimensional (continuous) representation of K there exists a hermitian scalar product on V such that the representation is irreducible.
3) Let $K \to \mathrm{GL}(V_i)$ be two finite dimensional irreducible unitary representations. They are unitary isomorphic if and only of the isomorphic in the usual sense.

Hence the classification of irreducible unitary representation of a compact group and the classification of finite dimensional irreducible representations is the same. So we can forget about the scalar products.

The representation theory of SU(n) is closely related to the theory of *rational* representations of $\mathrm{GL}(n, \mathbb{C})$. We have to consider polynomial functions f on $\mathrm{GL}(n, \mathbb{C})$. These are function which can be written as polynomials in the n^2 variables a_{ik}. More over a function f on $\mathrm{GL}(n, \mathbb{C})$ is called *rational* if there exists a natural number k such that $\det A^k f(A)$ is polynomial. We will use the following result without proof.

1.1 Proposition. *Every finite dimensional (continuous) representations $\pi : \mathrm{U}(n) \to \mathrm{GL}(V)$ extends to a rational representation. To finite dimensional representations of* U(n) *are isomorphic if and only if their rational extensions are isomorphic. The representation π is irreducible if an only if its rational extension is irreducible.* RatExt

Hence the classification of irreducible unitary representations of U(n) is the same as the classification of irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$. The classification of the irreducible rational representations can be given by their highest weight.

1.2 Theorem. *Let $\pi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ be an irreducible rational representation. There exists a one-dimensional subspace $W \subset V$ that is invariant under all upper triangular matrices and W is unique with this property. There exist integers* ClasRat

$$r_1 \ge r_2 \ge \cdots r_n$$

such that the action of diagonal matrices A with diagonal $a_1, \dots, a_n$ is given by

$$\pi(A)w = a_1^{r_1} \cdots a_n^{r_n} w \quad (w \in W).$$

This gives a bijection between the set of isomorphy classes of irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$ and the set of increasing tuples $r_1 \ge \cdots \ge r_n$ of integers.

We will not prove this result here. □

The tuple $(r_1 \ge \cdots \ge r_n)$ is called the highest weight and the elements of W are called highest weight vectors.

This theorem does not tell, how the representations of a given highest can can be constructed and, in particular, it does not tell the dimensions of the representations.

For us, the case $n = 2$ is of special importance. Let

$$l \in \{0, 1/2, 1, 3/2, \dots\}$$

be a non negative integral or half integral non-negative number. We consider the special V_l of all polynomial functions $P : \mathbb{C}^2 \to \mathbb{C}$ which are homogenous and of degree $2l$. So the dimension of V_l is $2l + 1$. We define a representation

$$\varrho_l : \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathrm{GL}(V_l)$$

by

$$(\varrho(g)P)(x) = P(g^{-1}x).$$

(We use the standard action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^2$ of linear algebra which is an action from the left.) The subspace W_l that is generated by the polynomial $P(x, y) = y$ is invariant under upper triangular matrices and it is the only one dimensional subspace with this property. From this one can deduce that ϱ_l is irreducible. The highest weight is $(0, 2l)$. More generally we can consider the representation $\det(g)^k \varrho_l(g)$. Its highest weight is $(k, k + 2r)$. These pairs exhaust all highest weights. Hence we have found all representations of $\mathrm{GL}(2, \mathbb{C})$ and as a consequence also of $\mathrm{U}(2)$.

We are more interested in $\mathrm{SU}(2)$. It is not difficult to show that every finite dimensional irreducible representation of $\mathrm{SU}(2)$ is the restriction of a representation of $\mathrm{U}(2)$. In this way one can prove the following result.

1.3 Theorem. *The restriction of ϱ_l to $\mathrm{SU}(2)$ is irreducible. Every finite dimensional irreducible representation is isomorphic to one and only one representation of this list.* SU1

2. The Lie algebra of the complex linear group of degree two

We need the notion of the complexification of a real vector space V. By definition this is $V_{\mathbb{C}} = V \times V$ as real vector space. The multiplication by i is given by

$$\mathrm{i}(a, b) = (-b, a).$$

This extends to an action of $\mathbb{C}$ on $V_{\mathbb{C}}$ through

$$(\alpha + \beta)x = \alpha x + \mathrm{i}\beta x \qquad (x \in V_{\mathbb{C}})$$

and this equips $V_{\mathbb{C}}$ with a structure as complex vector space We can embed V into $V_{\mathbb{C}}$ by $a \mapsto (a, 0)$ and if we identify V with its image than $V_{\mathbb{C}}$ is the direct sum of V and $\mathrm{i}V$. The following universal property holds. Let $f : V \to W$ be an $\mathbb{R}$-linear map into a complex vector space W. Then there exist a unique $\mathbb{C}$-linear extension $V_{\mathbb{C}} \to W$. Just map (a, b) to $f(a) + \mathrm{i}f(b)$.

We must give a warning. The vector space V might be a complex vector space in advance. Of course we can consider V as real vector space and then take its complexification. But the complex structures of V and $V_{\mathbb{C}}$ are different things. Recall that we defined $\mathrm{i}(a, b) = (-b, a)$. This is different from $(\mathrm{i}a, \mathrm{i}b)$. Hence we prefer to use the notation

$$J(a, b) = (-b, a).$$

One can consider the space $\tilde{\mathbb{C}}$ of operators $\alpha\,\mathrm{id} + \beta J$ with reals α, β. This is a ring and even more it is a field which is isomorphic to the usual field of complex numbers. This “copy” of the field of complex numbers acts now on $V_{\mathbb{C}} = V \times V$ and in this way we get that $V_{\mathbb{C}}$ is a field over $\tilde{\mathbb{C}}$. We apply this to the Lie-algebra $\mathfrak{g}$ of all *complex* 2×2-matrices with trace zero. We have to consider its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$. We extend the Lie-bracket to ${}_{\mathbb{C}}$ by the formula

$$[(A_1, A_2), (B_1, B_2)] := ([A_1, B_1] - [A_2, B_2], [A_1, B_2] + [A_2, B_1]).$$

So we have $[(A, 0), (B, 0)] := ([A, B], 0)$ and we have

$$[J(A_1, A_2), (B_1, B_2)] = J[(A_1, A_2), (B_1, B_2)] = [(A_1, A_2), J(B_1, B_2)].$$

So the extended Lie-bracket is the $\tilde{\mathbb{C}}$-bilinear extension of the Lie bracket on $\mathfrak{g}$ if we consider it embedded into $\mathfrak{g}_{\mathbb{C}}$ by $A \mapsto (A, 0)$. Now we consider

$$\mathfrak{g}_{\mathbb{C}}^{+} = \{(A, \mathrm{i}A), \quad A \in \mathfrak{g}\}, \qquad \mathfrak{g}_{\mathbb{C}}^{-} = \{(A, -\mathrm{i}A), \quad A \in \mathfrak{g}\}.$$

Here iA refers of course to the internal multiplication of $\mathfrak{g}$. Every element $(X, Y) \in_{\mathbb{C}}$ can be written uniquely in the form $(X, Y) = (A, \mathrm{i}A) + (B, -\mathrm{i}B)$. In other words, we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{+} \oplus \mathfrak{g}_{\mathbb{C}}^{-}.$$

It is easy to check that $\mathfrak{g}_{\mathbb{C}}^{\pm}$ are invariant under the Lie brackets, for example

$$[(A, \mathrm{i}A), (B, \mathrm{i}B) = 2([A, B], \mathrm{i}[A, B]).$$

Both are invariant under the operator J. Hence they are vector spaces over $\tilde{\mathbb{C}}$. To be more precise: they are $\tilde{\mathbb{C}}$-sub vector spaces of $\mathfrak{g} \times \mathfrak{g}$. This shows more, namely that the map

$$\mathfrak{g}_{\mathbb{C}}^{+} \xrightarrow{\sim} \mathfrak{g}, \quad (A, \mathrm{i}A) \longmapsto \sqrt{2}A$$

is a real linear isomorphism which respects the Lie brackets. Similarly

$$\mathfrak{g}_{\mathbb{C}}^{-} \xrightarrow{\sim} \mathfrak{g}, \quad (A, -\mathrm{i}A) \longmapsto \sqrt{2}A$$

respects the Lie brackets. We also note that the bracket between an element of $\mathfrak{g}_{\mathbb{C}}^{+}$ and $\mathfrak{g}_{\mathbb{C}}^{-}$ is zero. So we proved the following Lemma.

2.1 Lemma. *The map* LieCZ

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}_{\mathbb{C}}, \quad (A, B) \longmapsto \frac{1}{\sqrt{2}}\big((A, \mathrm{i}A) + (B, -\mathrm{i}B)\big),$$

is an isomorphism of real vector spaces. The Lie bracket of $\mathfrak{g}_{\mathbb{C}}$ *corresponds on the left hand side to the bracket*

$$[(A_1, A_2), (B_1, B_2)] := [A_1, A_2] + [B_1, B_2].$$

We consider now also the Lie-algebra $\mathfrak{g}_0$ consisting of all real 2×2-matrices of trace 0. This is the algebra which we considered earlier. The algebra $\mathfrak{g}_{\mathbb{C}}$ which we consider now can be considered as the complexification of $\mathfrak{g}_0$. We restrict now the isomorphism of Lemma 3.2.1 to $_0 \times \mathfrak{g}_0$. Then we obtain the following Lemma.

2.2 Lemma. *The map* LieCh

$$\mathfrak{g}_0 \times \mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{g}_{\mathbb{C}}, \quad (A, B) \longmapsto \frac{1}{\sqrt{2}}\big((A, \mathrm{i}A) + (B, -\mathrm{i}B)\big),$$

is an injective real linear map which preserves the Lie brackets, where the left hand side is equipped with the bracket

$$[(A_1, A_2), (B_1, B_2)] := [A_1, A_2] + [B_1, B_2].$$

Denote by $\mathfrak{h}$ *the image of this map. Then we have*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus J\mathfrak{h}.$$

The proof is given by trivial computation. □

So this shows the following. One can identify $\mathfrak{g}_{\mathbb{C}}$, which, by definition, was constructed as complexification of $\mathfrak{g}$, also with the complexification of $\mathfrak{g}_0 \times \mathfrak{g}_0$.

Now we consider a (complex) algebra $\mathcal{A}$ and a real linear Lie algebra homomorphism

$$f : \mathfrak{g} \longrightarrow \mathcal{A}.$$

"Lie homomorphism" means that $f([A, B] = f(A)f(B) - f(B)f(A)$. We can extend it to a $\mathbb{C}$-linear Lie homomorphism $f : \mathfrak{g}_{\mathbb{C}} \to \mathcal{A}$. Here $\mathbb{C}$-linear refers of course to the complexification complex structure of $_{\mathbb{C}}$ which means

$$f(JA) = \mathrm{i}f(A).$$

This gives us a Lie homomorphism

$$\mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathcal{A}.$$

Restricting it to $\mathfrak{g}_0$ by $A \mapsto (A, 0)$ or $(0, A)$ we get two Lie homomorphisms

$$f^{\pm} : \mathfrak{g}_0 \longrightarrow \mathcal{A}.$$

Recall that such a homomorphism produces a certain Casimir element in $\mathcal{A}$. It has the property that it commutes with the image of $\mathfrak{g}_0$. Hence we get now two Casimir elements ω^+ and ω^-. From the Lemmas 2.1 and 2.2 we see that the two Casimir elements commute with the image of $_{\mathbb{C}}$ in $\mathfrak{A}$.

2.3 Propostition. *Let $\mathfrak{g} \to \mathcal{A}$ be a real linear Lie homomomorphism. We extend it by complex linearity to $\mathbb{Z}_{\mathbb{C}}$ and restrict it in two ways to Lie homomorphism $\mathfrak{g}_0 \to \mathcal{A}$ (Consider the two obvious embeddings of $\mathfrak{g}_0$ into $\mathfrak{g}_0 \times \mathfrak{g}_0$ and consider the embedding of the latter into $\mathfrak{g}_{\mathbb{C}}$. These Lie homomorphisms produce Casimir elements $\omega^{\pm} \in \mathcal{A}$. They commute with the image of $\mathfrak{g}_{\mathbb{C}}$.* TwoCas

3. Representations of the complex special linear group of degree two

From now on we use the notations

$$G = \mathrm{SL}(2, \mathbb{C}), \qquad K = \mathrm{SU}(2).$$

We consider irreducible unitary representations $\pi : G \to \mathrm{GL}(H)$. We consider the restriction of K and decompose the representation into irreducibles under K. The basic result is now:

3.1 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ by an irreducible unitary representation. In the restriction of π to K each irreducible representation of K occurs with multiplicity ≤ 1.* MulOne

We omit the proof. □

We denote by $\mathcal{H} = H(K)$ the algebraic sum of the K-irreducible subspaces of H. It can be shown that they consist of differentiable ((even analytic) vectors such that the derived representation

$$d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{H})$$

can be defined through the same formula as in the $\mathrm{SL}(2,\mathbb{R})$-case. Then we can consider the Casimir operators $\omega^{\pm} \in \mathrm{End}(\mathcal{H})$. Similar to the $\mathrm{SL}(2,\mathbb{R})$-case it can be shown that both act be multiplication with constants $\mu^{\pm}$. These are basic invariants of the representations. We can define an other invariant $l \in \{0, 1/2, 1, \dots\}$. It is the smallest l such that the representation $\varrho : K \to \mathrm{GL}(V_l)$ occurs in $\pi|K$.

3.2 Theorem. *An irreducible unitary representation is determined by its paramaters l, μ^+, μ^- up to unitary isomorphism. The parameters $\mu^{\pm}$ are real and satisfy the relations* ThSC

$$\mu_2^2 = 32l^2(\mu_1 + 8l^2 - 8), \quad 32(l+1)^2(\mu_1 + 8l^2 + 16l) - \mu_2^2 > 0.$$

The proof cannot be given here. □

The point is now that in the cases which are described in the theorem, all paramters under the described constraints can be actually realized by an irreducible unitary representation of $\mathfrak{g}$. Similar to the $\mathrm{SL}(2,\mathbb{R})$-case, it is better to introduce a new parameter s be the definition

$$s^2 = (\mu_1 + 8l^2 - 8)/8.$$

This means

$$\mu_1 = 8(s^2 + 1 - l^2), \quad \mu_2^2 = (16ls)^2.$$

In the case $l \neq 0$ we can fix s such that

$$\mu_2 = 16ls \qquad (l \neq 0).$$

In the case $s = 0$ this makes no sense. So in this case wa have to be satisfied with the fact that s is only determined up to sign.

Now we have to check for which s the inequaltiy in Theorem 3.2 us satisfied. Obviously it is satisfied if s is a real number. We call the triples l, μ^+, μ^- which came from real s the prinicipal series. But that ist all. In the case $l = 0$ one This parameter is defined up to the sign. With this parameter we can also take $s = it$ where $t \in (-1, 1)$. This is called the complementary series.

3.3 Theorem. ***Principal Series.*** ClssTH
For every $l \in \{0, 1/2, 1, \ldots\}$ and for any ***real*** *s there exists an irreducible unitary representation $\pi_{l,s}$ which produces the parameters (l, μ^+, μ^-) where*

$$\mu_1 = 8(s^2 + 1 - l^2), \quad \mu_2 = 16ls.$$

The parameter s is uniquely determined if $l \neq 0$ and uo to the sign if $l = 0$.

Complementary Series.
The same statement is true for $l = 0$ and $s = \mathrm{i}t$, $t \in (-1, 1)$. Here s is also determined.

Every unitary irreducible representation of $\mathrm{GL}(2, \mathbb{C})$ is unitary isomorphic to a representation of these two lists.

In the following we will describe the realization of the principal series.

Chapter IV. Mackey's theory of the induced representation

1. Induced representations, simple case

The basic idea of induced representations is easy to explain. Let $P \subset G$ be a subgroup of a finite group and $\sigma : P \to \mathrm{GL}(H)$ a representation of the subgroup. We consider the space $\mathrm{Ind}(\sigma)$ of all functions $f : G \to H$ with the property

$$f(px) = \sigma(p)f(x) \quad \text{for} \quad p \in P,\ x \in G.$$

Then G acts by right translation on $\mathrm{Ind}(\sigma)$. Assume that G is a locally compact group and that P is a closed subgroup. We want to modify this construction in such a way that we get – for certain σ – a *unitary* induced representation. An example was already given by the construction of the principal series in Chap. I, Sect. 7. Here $G = \mathrm{SL}(2, \mathbb{R})$ and P is the subgroup of all upper triangular matrices with positive diagonal, σ was the one dimensional unitary representation given by the character $\sigma(p) = a^{1+s}$ where $\mathrm{Re}\, s = 0$.

Already in this case we had to deal with the problem is that the condition $\Delta_G|P = \Delta_P$ may be false so there is no G-invariant measure on $P\backslash G$.

Before we go to the general case we make a very restrictive assumption which was satisfied in the example of the principal series. We assume that there exists a closed subgroup $K \subset G$ be a closed subgroup of the locally compact group G such that the multiplication map $P \times K \to G$ is a topological map. We assume that G and K are unimodular but we do not assume that P is unimodular. Let Δ be the modular function of P.

1.1 Lemma. *Let $y \in G$. We consider the (continuous) maps $\alpha : K \to K$ and $\beta : K \to P$ which are defined by $ky = \beta(k)\alpha(k)$. Then for each $f \in \mathcal{C}_c(K)$ the formula* TmIz

$$\int_K f(\alpha(k))\Delta(\beta(k))dk = \int_k f(k)dk$$

holds.

Proof. This is a generalization of Lemma I.7.1. The same proof works. □

Now we can give a straight forward generalization of the principal series. Let $\sigma : P \to \mathrm{GL}(H)$ be a *unitary* representation of P. We consider functions

$f : G \to V$ with the transformation property

$$f(py) = \Delta(p)^{1/2}\sigma(p)f(y), \qquad p \in P,\ y \in G.$$

(it is essential that we do not induce σ directly but modify it with the factor $\Delta(p)^{1/2}$.) Such a function is determined by its restriction to K and every function on K can be extended to a function with this transformation property on G. The group G acts by translation from the right on the space of functions with this transformation property. We can this consider as an action of G on the space of all functions $f : K \to V$.

1.2 Proposition. *We assume that $G = PK$ and that G and K are both unimodular. Let $\sigma : P \to \mathrm{GL}(V)$ be a unitary representation. The group G acts on functions $f : G \to V$ with the transformation property* IndRep

$$f(py) = \Delta(p)^{1/2}\sigma(p)f(y), \qquad p \in P,\ y \in G$$

*by translation from the right. These functions can be identified with functions $f : K \to V$. Zero functions on K are transformed into zero functions and square integrable functions into square integrable ones. This induces a **unitary** representation π of G on $L^2(K, V, dk)$.*

The proof is the same as that of Proposition I.7.2.

2. Induced representations, the general case

Unfortunately this construction which we gave in Sect. 2 is not good enough. We want to give up the existence of a decomposition $G = KP$. We simply assume that $P \subset G$ is a closed subgroup of a locally compact group.

The following procedure to overcome this difficulty is due to Mackey. Since we have do differ between left- and right-invariant measures, we will denote by $d_l x$ a left invariant measure on G and by $d_r p$ a right invariant measure on P.

2.1 Assumption. *There exists a function $q : G \to \mathbb{R}_{>0}$ which is measurable for any Radon measure on G such that q and q^{-1} are locally bounded and with the property* AssuC

$$q(px) = \frac{\Delta_P(p)}{\Delta_G(p)} q(x).$$

There is an important case where the existence of a function q is trivial.

2.2 Remark. *Let $P, K \subset G$ we closed subgroups of a locally compact group such that the map* PKAss

$$P \times K \xrightarrow{\sim} G, \quad (p,k) \longmapsto pk,$$

is topological. Then the function

$$q(pk) = \frac{\Delta_P(p)}{\Delta_G(p)}$$

satisfies the Assumption 2.1.

In this case the function q is continuous. The essential point that the Assumption 2.1 is satisfied in much more general situations (with functions q which may be nor continuous).

Let us assume for example the following:

There exists an open non-empty subset $U \subset P\backslash G$ and a continuous section $s : U \to G$.

(Section means that $s(a)$ is a representative of the coset $a \in P\backslash G$.) Let $\tilde{U}$ be the inverse image of U in G and $\pi : \tilde{U} \to U$ the natural projection. We can consider the continuous function

$$q_U : \tilde{U} \to \mathbb{R}_{>0}, \quad q_U(x) = \frac{\Delta_G}{\Delta_P}(xs(\pi(x))^{-1}).$$

Then q_U has the desired transformation on $\tilde{U}$. Taking translates we can cover $P\backslash G$ with sets U. Since we have countable basis of the topology we can write

$$P\backslash G = U_1 \cup U_2 \cup \cdots$$

such that in each inverse image $\tilde{U}_i$ a function $q_i = q_{U_i})$ with such a property exist. We want to glue the q_i and do this in the most simple way. We consider the *disjoint* decomposition

$$P\backslash G = B_1 \dot{\cup} B_2 \dot{\cup} \dots \quad \text{where} \quad B_n = U_n - (U_1 \cup \ldots \cup U_{n-1}).$$

We now define q such that its restriction to B_i is q_i. This is a measurable function for any Radon measure, since the sets B_i are measurable sets. (They are Borel sets).

The assumption that a local continuous section s exists is weak. It is always satisfied in the context of Lie groups. The reason is that for a Lie group G there is a vector space $\mathfrak{g}$ (the Lie-algebra) and a surjective map $\mathfrak{g} \to G$ which is a local homeomorphism close to the origin. Even more, there exists a subspace $\mathfrak{p} \subset \mathfrak{g}$ which plays the same role for P. Consider a decomposition of vector spaces $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{a}$. Then $\mathfrak{a} \to P\backslash G$ is a local homeomorphism close to the origin. Now the existence of a local section is clear, it just corresponds to the natural imbedding of $\mathfrak{a}$ into $\mathfrak{g}$.

This argument applies in all situations which we need. In the following we take Assumption 2.1 to be granted. We need a generalization of the construction of quotient measures (Proposition I.3.3).

2.3 Proposition. *Assume that $P \subset G$ is a closed subgroup of a locally compact group and that q is a function as in Assumption 2.1. Then there exists a unique Radon measure $d\bar{x}$ on $P\backslash G$ (depending on q) such that the formula* QuotGen

$$\int_G f(x)q(x)d_lx = \int_{P\backslash G}\left[\int_P f(px)d_rp\right]d\bar{x}$$

holds for $f \in \mathcal{C}_c(G)$. Here d_rp denotes a right invariant measure on P. and d_lx a left invariant measure on G (suitably normalized).

The proof is the same as that of the existence of the quotient measure, Proposition 1.3.3 (which we did not give in full detail). We just mention the essential fact that the inner integral is left invariant as function of x, since d_rp has been taken to be right invariant. We also mention that for $f \in \mathcal{C}_c(G)$, the function $f(x)q(x)$ is an integrable function on G (with respect to any Radon measure).

Usually we will write dx instead of $d\bar{x}$ as long this is not expected to cause confusion.

This measure on $P\backslash G$ is not invariant under the action of G. But it has still the weaker problem that the space of zero functions is invariant under (right) translation with elements of G.

One can use this measure to define the induced representation of a unitary representation $\sigma : P \to \mathrm{GL}(H)$.

2.4 Definition and Remark. *Assume that $P \subset G$ is a closed subgroup of a locally compact group and that q is a function as in Assumption 2.1. Let dx be the corresponding measure on $P\backslash G$. Let $\sigma : P \to \mathrm{GL}(H)$ be a unitary representation. Consider the space of all measurable functions $f : G \to H$ with the property $f(px) = \sigma(p)f(x)$ and such that $\|f(x)\|_\sigma^2$ is integrable considered as function on $P\backslash G$. The quotient of this space by the subspace of all functions, such that $\|f(x)\|_\sigma^2$ is a zero function (considered on $P\backslash G$), is a Hilbert space $H(\sigma)$ with the hermitian inner product* IndRe

$$\langle f, g\rangle = \int_{P\backslash G}\langle f(x), g(x)\rangle_\sigma dx.$$

The group G acts on it by means of the modified translation from the right: for $g \in G$ the operator R_g is defined by

$$(R_gf)(x) = f(xg)\sqrt{\frac{q(x)}{q(xg)}}.$$

This is a unitary representation, called the (unitary) induced representation of σ to G. It is independent of the choice of q up to unitary isomorphism.

3. Stone's theorem

We study unitary representations of the additive group $\mathbb{R}^n$ which are not necessarily irreducible. We give an example. Let (X, dx) be a Radon measure and $f : X \to \mathbb{C}$ be a measurable and bounded function. Then we can define the multiplication operator

$$m_f : L^2(X, dx) \longrightarrow L^2(X, dx), \quad g \longmapsto fg.$$

This is a bonded linear operator. A bound is given by $\sup_{x\in X} |f(x)|$. The adjoint of m_f is $m_{\bar{f}}$. Hence m_f is self adjoint for real f and unitary if $|f(x)| = 1$ for all x. If f is the characteristic function of a measurable set, we have $m_f^2 = m_f$. This means that $P = m_f$ is an orthogonal projection. This implies that there exists an orthogonal decomposition $H = H_1 \oplus H_2$ such that $P(h_1 + h_2) = h_2$. Just take for H_1 the kernel of P and for H_2 its orthogonal complement. This is the image of P.

If f_n is a sequence of uniformly bounded functions that converges pointwise to f then m_{f_n} converges pointwise to m_f.

Now we assume that $f : X \to \mathbb{R}^n$ is a measurable (not necessarily bounded) function. Then we can consider for each $a \in \mathbb{R}^n$ the bounded and measurable function

$$x \longmapsto e^{\mathrm{i}\langle a, f(x)\rangle} \qquad (\langle a, b\rangle = a_1b_1 + \cdots a_nb_n).$$

We denote by $U(a)$ the multiplication operator for this function. Obviously this is and unitary operator and moreover $a \longmapsto U(a)$ is an unitary representation. We call it the multiplication representation related to f.

3.1 Stone's theorem. *Let $U : \mathbb{R}^n \to \mathrm{GL}(H)$ be a unitary representation.* ST
Then there exists a Radon measure (X, dx) and a continuous function $f : X \to \mathbb{R}^n$ and a Hilbert space isomorphism $\sigma : H \to L^2(X, dx)$ such that the transport of U to $L^2(X, dx)$ equals the multiplication representation related to f.

The space (X, dx) is not uniquely determined.

In the following we use the notations of Stone's theorem. We consider a bounded function $\varphi : \mathbb{R}^n \to \mathbb{C}$. We always assume that $\varphi \circ f$ is measurable with respect to dx. This is for example the case when φ is continuous. Another case which we will use is that φ is the characteristic function of a Borel set $\mathcal{B} \subset \mathbb{R}^n$, since then $\varphi \circ f$ is the characteristic function of $f^{-1}(\mathcal{B})$ which is also a Borel set. Then we can consider the multiplication operator $m_{\varphi\circ f}$. We use the isomorphism in Theorem 4.3.1 to transport it to a bounded linear operator which we denote by same letter

$$M_\varphi = m_{\varphi\circ f} : H \longrightarrow H.$$

3.2 Remark. *The operator M_φ depends only on the representation $U(a)$ and* Rrf
the function φ (and not on the choice of (X, dx) and the isomorphism σ).

We omit the proof since we do not need in what follows. □

In the special case that $\varphi(x) = e^{2\pi \mathrm{i}\langle a,x\rangle}$, we get back $M_\varphi = U(a)$. If φ is the characteristic function of a Borel set $\mathcal{B} \subset \mathbb{R}^n$, we denote this operator by

$$M(\mathcal{B}) = M_\varphi : H \longrightarrow H.$$

3.3 Lemma. *We use the notations of Theorem 3.1. The map $\mathcal{B} \to M(\mathcal{B})$ from Borel sets to orthogonal projection operators on H has the following properties.* LBb

1) $M(\emptyset) = 0$.
2) $M(\mathbb{R}^n) = \mathrm{id}$.
3) *Let $\mathcal{B} = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \dots$ by a disjoint union of countable many Borel sets. Then*

$$M(\mathcal{B}) = \sum M(\mathcal{B}_n)$$

(pointwise convergence of operators).

Usually M is called a the "spectral measure" of the representation U.

4. Mackey's theorem

Let G be a locally compact group and let L, A be two closed subgroups. We assume that A is an abelian normal subgroup and that the map

$$L \times A \longrightarrow G, \quad (h,a) \longmapsto ha,$$

is topological. Then L acts on A by conjugation gag^{-1}. We are interested in continuous characters

$$\alpha : A \longrightarrow S^1.$$

They form a group A'. Two characters α, β are called equivalent if there exists $g \in L$ such that

$$\beta(a) = \alpha(gag^{-1}).$$

The equivalence classes with respect to this equivalence relation are called orbits. For a character α we consider the group

$$L_\alpha = \{\, g \in G; \quad \alpha(gag^{-1}) = \alpha(g) \,\}.$$

This group depends essentially only on the orbit of α. This means the following. Let $\beta(a) = \alpha(g\alpha g^{-1})$ be another character in the orbit. Then

$$L_\beta = gL_\alpha g^{-1}.$$

The subgroups of L of the form L_α are called *small subgroups.*

Now we consider an irreducible unitary representation

$$\varrho : G_\alpha \longrightarrow \mathrm{GL}(H).$$

We can induce this representation to an unitary representation of G. Mackey's theorem states that – under a certain assumption – this is an irreducible unitary representation of G and that each irreducible unitary representation is isomorphic to such one.

We now explain this assumption. For this we have to know that A' carries a structure of a locally compact group as well. We will only explain this for the group $A = \mathbb{R}^n$, since this is the only case we will use.

4.1 Lemma. *Let $(\cdot,\cdot)$ be any non-degenerated bilinear form on $\mathbb{R}^n$. Every continuous character on $\mathbb{R}^n$ is of the form* CharRn

$$L : \mathbb{R}^n \longrightarrow S^1, \quad L(x) = e^{\mathrm{i}(a,x)} \qquad (a \in \mathbb{R}^n).$$

This gives an isomorphism

$$\mathbb{R}^n \longmapsto (\mathbb{R}^n)', \quad a \longmapsto L.$$

We use this isomorphism to equip $(\mathbb{R}^n)'$ with a topology. Of course this topology is independent on the choice of $(\cdot,\cdot)$. Now we can formulate the assumptions for Mackey's theorem.

4.2 Assumption. *There exists a closed subset in A' which intersects with each orbit in exactly one point.* AssB

This means that we can choose from each orbit a representative in some regular way.

Now we can formulate Mackey's theorem.

4.3 Mackey's thoerem. *The induced representations on G from irreducible unitary representations of inner groups are unitary and irreducible. Each irreducible unitary representation of G is isomorphic to one of this type.* MackT

Chapter V. Unitary representations of the Poincaré group

1. The Lorentz group

The Minkowski space of dimension $n+1$ is the vector space $\mathbb{R}^{n+1}$ that has been equipped with the symmetric bilinear form

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots x_{n+1}y_{n+1}.$$

A vector is called time-like if $\langle x, x\rangle = 0$. The set of all time like vectors consists of two connected cones. One of them is define by $x_1 > 0$. We call this the future cone.

The Lorentz group is the subgroup of $\mathrm{GL}(\mathbb{R}^{n+1})$ that preserves this form, $\langle gx, gy\rangle = \langle x, y\rangle$. If one identifies $\mathrm{GL}(\mathbb{R}^{n+1})$ with $\mathrm{GL}(n+1, \mathbb{R})$ in the usual manner, then this means

$$A'JA = J \quad \text{where} \quad J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We denote the Lorentz group by $\mathrm{O}(n, 1)$. We always assume $n > 0$. There are two important subgroups. The first is the special orthogonal group $\mathrm{SO}(n, 1)$ which consists of all elements with determinant one. The second is the subgroup $\mathrm{O}^+(n, 1)$ that preserves the future cone. Since time like vectors are mapped to time like vectors, it is sufficient to know that the vector $(1, 0, \ldots, 0)$ is mapped to a vector a with $a_1 \geq 0$. For the matrix A this means that $a_{11} > 0$. Hence we have seen that the set of all matrices in the Lorentz group with this property build a group. The elements of this group are called loxodromic. The matrix J is in the Lorentz group and has determinant -1. This shows

$$\mathrm{O}(n, 1) = \mathrm{SO}(n, 1) \cup \mathrm{SO}(n, 1)J.$$

The negative of the unit matrix E is not loxodromic. Hence we see

$$\mathrm{O}(n, 1) = \mathrm{O}^+(n, 1) \cup \mathrm{O}^+(n, 1)(-E).$$

We use the notation

$$\mathrm{SO}^+(n,1) = \mathrm{O}^+(n,1) \cap \mathrm{SO}(n,1).$$

We see

$$\mathrm{O}(n,1) = \mathrm{SO}^+(n,1) \,\cup\, \mathrm{SO}^+(n,1)J \,\cup\, \mathrm{SO}^+(n,1)(-E) \,\cup\, \mathrm{SO}^+(n,1)(-J).$$

It can be shown that $\mathrm{SO}^+(n,1)$ is open in $\mathrm{O}(n,1)$ and connected. Hence $\mathrm{O}(n,1)$ has 4 connected components.

For small n one can find different descriptions. We start with $\mathrm{O}(2,1)$. For this we consider the vector space $\mathcal{X}$ of all skew symmetric real 2×2-matrices

$$X = \begin{pmatrix} x_2 & x_1 \\ -x_1 & x_3 \end{pmatrix}.$$

Their determinant is $-x_1^2 + x_2^2 + x_3^2$. We identify $\mathcal{X}$ with $\mathbb{R}^3$ in the obvious way. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{X}$ through $(A,X) \longmapsto AXA'$. For given A this transformation can be considered as element of $\mathrm{GL}(3,\mathbb{R})$. The above formula for the determinant shows that it is in $\mathrm{O}(3,\mathbb{R})$. From the Iwasawa decomposition one can see that $\mathrm{SL}(2,\mathbb{R})$ is connected. Hence we constructed a homomorphism $\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(3,1)$.

1.1 Proposition. *The homomorphism* SpinZw

$$\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(3,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(3,1)$ *has precisely two inverse images which differ only by the sign.*

We skip the proof of the surjectivity. □

Proposition 1.1 is only a special case of a general result. For each n there exists connected locally compact group G and a continuous surjective homomorphism $G \to \mathrm{SO}^+(n,1)$ such that each element of the image has precisely two pre-images. This group is (in an obvious sense) essentially unique and called the spin covering. The usual notation is $\mathrm{Spin}(n,1)$ for this group. So $\mathrm{Spin}(2,1) = \mathrm{SL}(2,\mathbb{C})$ We don't give this (not quite trivial construction) in the general case and treat only the case $n=3$ which is fundamental for physics.

For the construction of $\mathrm{Spin}(3,1)$ ce consider the space of all hermitian 2×2-matrices

$$H = \begin{pmatrix} h_0 & h_1 \\ \bar h_1 & h_2 \end{pmatrix}.$$

We identify $\mathcal{H}$ with $\mathbb{R}^4$ through

$$H \longmapsto \Big(\frac{h_0+h_2}{2}, \frac{h_0-h_2}{2}, \operatorname{Re} h_1, \operatorname{Im} h_1\Big).$$

Then we have
$$\det H = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{H}$ through $(A,H) \longmapsto AH\bar{H}'$ It preserves the determinant. Hence we obtain a Lorentz transformation. It can be shown that $\mathrm{SL}(2,\mathbb{C})$ is connected two. Hence we get a homomorphism

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(1,3).$$

1.2 Proposition. *The homomorphism* SpinDr

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(3,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(3,1)$ *has precisely two inverse images which differ only by the sign.*

This allows us to write $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$.

The existence of spin coverings is not tied to signature $(n,1)$. For example we can consider the Euclidian orthogonal group $\mathrm{O}(3,\mathbb{R})$. Recall that $\mathrm{O}(n,\mathbb{R})$ consists of all $A \in \mathrm{GL}(n,\mathbb{R})$ with the property $A'A = E$. This is a closed subgroup. The rows and columns have Euclidean length 1. Hence $\mathrm{O}(n,\mathbb{R})$ is a compact group (in contrast to the Lorentz group!). The subgroup $\mathrm{SO}(n,\mathbb{R})$ of elements of determinant one is called the special orthogonal group. One can show that it is connected. The group $\mathrm{O}(n,\mathbb{R})$ can be embedded into the Lorentz group $O(n,1)$ by means of

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

We consider this in the case $n = 3$. We can consider the inverse image in $\mathrm{SL}(2,\mathbb{C})$. One can check that this inverse image is the special unitary group $\mathrm{SU}(2)$. Recall that The unitary group $\mathrm{U}(n)$ is the subgroup of all $A \in \mathrm{GL}(n,\mathbb{C})$ with the property $\bar{A}'A = E$. This is a compact group. The special unitary group is the subgroup of all A with $\det A = 1$. One can show that it is connected.

1.3 Proposition. *The homomorphism* SpinDef

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3,\mathbb{R})$$

is continuous and surjective. Each element of $\mathrm{SO}(3,\mathbb{R})$ *has precisely two inverse images which differ only by the sign.*

Hence we can write $\mathrm{Spin}(3,\mathbb{R}) = \mathrm{SU}(2,\mathbb{C})$.

2. The Poincaré group

In the following we call $\mathrm{O}(n,1)$ the *homogeneous Lorentz group*. The *inhomogeneous Lorentz group* is the set of all transformations of $\mathbb{R}^{n+1}$ of the form

$$v \longmapsto A(v) + b$$

where A is a Lorentz transformation and $b \in \mathbb{R}^{n+1}$. This group can be identified with the set $\mathrm{O}(n,1) \times \mathbb{R}^{n+1}$. The group law then is

$$(g,a)(h,b) = (gh, a + gb).$$

We write for the inhomogeneous Lorentz group simply

$$\mathrm{O}(n,1)\mathbb{R}^{n+1}.$$

We want to define also a "spin covering". For this we have to consider the action of $\mathrm{Spin}(n,1)$ on $\mathbb{R}^{n+1}$ which is defined by means of the natural homomorphism $\mathrm{Spin}(n,1) \to \mathrm{O}(n,1)$ and the natural action of $\mathrm{O}(n,1)$ on $\mathbb{R}^{n+1}$. We write this action simply in the form $(g,v) \mapsto gv$. The *Poincaré group* $P(n)$ is the set

$$P(n) = \mathrm{Spin}(n,1) \times \mathbb{R}^{n+1}$$

together with the group law

$$(g,a)(h,b) = (gh, a + gb).$$

It is clear that this is a group and that the natural map

$$P(n) \longrightarrow \mathrm{O}(n,1)\mathbb{R}^{n+1}$$

(spin covering on the first factor and identity on the second factor) is a homomorphism. This image is $\mathrm{SO}^+(n,1)\mathbb{R}^{n+1}$ and each element has two inverse images.

3. The spectral theorem

We have to consider the space $\mathcal{C}_c^\infty(\mathbb{R})$ of infinitely many differentiable complex valued functions on the real line. By a "Radon measure on $\mathcal{C}_c^\infty(\mathbb{R})$" we understand a $\mathbb{C}$-linear map $I : \mathcal{C}_c^\infty(\mathbb{R}) \to \mathbb{C}$ with the properties $I(\bar{f}) = \overline{I(f)}$ and $I(f) \geq 0$ if $f \geq 0$. It is easy to show that such an I extends uniquely to a Radon measure (on $\mathcal{C}_c(\mathbb{R})$). This follows from the fact that each $f \in \mathcal{C}_c(\mathbb{R})$ is the uniform limit of a sequence $f_n \in \mathcal{C}_c^\infty(\mathbb{R})$ whose supports are contained in a joint compact set.

A function $f : \mathbb{R} \to \mathbb{C}$ is called rapidly decreasing if Pf is bounded for all polynomials P. If f is measurable and rapidly decreasing, the (usual Lebesgue) integral

$$\int_{-\infty}^{\infty} f(t)dt$$

exists. A function is called tempered (or a Schwartz function) if it is infinitely often differentiable and if all derivatives are rapidly decreasing. The space of all tempered functions is called by $S(\mathbb{R})$. For tempered functions f the Fourier transform

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}tx} f(t)dt.$$

exists. It is easy to show (using partial integration) that it is tempered again. In particular, the Fourier transform of a C^∞-function with compact support is tempered.

3.1 Fourier inversion theorem. *The map $S(\mathbb{R}) \to S(\mathbb{R})$, $f \mapsto \hat{f}$, is an isomorphism of vector spaces. It extends to an isomorphism of Hilbert spaces* `Fit`

$$\mathcal{F} : L^2(\mathbb{R}, dt) \longrightarrow L^2(\mathbb{R}, dt)$$

where dt means the standard Lebesgue measure on the line. Moreover, one has

$$\mathcal{F}\mathcal{F}(f)(x) = f(-x).$$

Let (X, dx) be Radon measure and let $f : X \to \mathbb{C}$ be a bounded continuous function. Then we can define a bounded and linear operator

$$L^2(X, dx) \longrightarrow L^2(X, dx), \quad g \longmapsto fg.$$

In the case $\bar{f}f = 1$ this operator is unitary.

3.2 Spectral theorem. *Let $U : \mathbb{R} \to \mathrm{Un}(H)$ a unitary representation of the additive group $\mathbb{R}$ on a Hilbert space. Then there exists a Radon measure (X, dx) and a continuous function $f : X \to \mathbb{R}$ such that the representation U is equivalent to the representation* `SpT`

$$\tilde{U} : \mathbb{R} \longrightarrow \mathrm{Un}(L^2(X, dx), \quad \tilde{U}(t)(g) = e^{itf}g.$$

Equivalence means of course that there exists a Hilbert space isomorphism $W : L^2(\mathbb{R}, dx) \xrightarrow{\sim} H$ with the property $\tilde{U}(t) = W^{-1}U(t)W$.

We first treat a reduction of the spectral theorem to a special case. Let $\pi : G \to \mathrm{GL}(E)$ be a continuous representation. A vector a is called cyclic if the subspace generated by all $\pi(g)a$ is dense in E. This means that E is the only closed invariant subspace that contains a. The representation is (topologically) irreducible if and only if each non zero vector is cyclic. The existence of a cyclic vector is a much weaker condition.

When a cyclic vector exists, then the spectral theorem can be sharpened slightly as follows.

3.3 Proposition. *Let $U : \mathbb{R} \to \mathrm{Un}(H)$ a unitary representation of the additive group $\mathbb{R}$ on a Hilbert space. Assume that a cyclic vector exists. Then in the spectral theorem we can take $X = \mathbb{R}$ (and dx some Radon measure) and $f(t) = t$.* SpC

We first show that the general spectral theorem follows from Proposition 3.3 and hence after we prove the proposition.

Proposition 3.3 implies the spectral theorem 3.2. We claim the following.

Every unitary representation $\pi : G \to \mathrm{Un}(H)$ has the following property. H can be written as direct Hilbert sum of a finite or countable set of sub Hilbert spaces H_i which are invariant and such that each of them admits a cyclic vector.

This can be proved by a standard argument using Zorn's lemma. We leave the details to the reader. Such a decomposition is not at all unique. Hence one should not over emphasize its meaning.

The Radon measure that is used for the spectral theorem of (π, H) is the direct sum of the Radon measures for the single H_i. We explain briefly the notion of the direct sum. Let (X_i, dx_i) be a finite or countable collection of Radon measures. Then one defines their direct sum as follows. One takes the disjoint union X of the X_i. This is the set of all pairs (x, i), $x \in X_i$. There is a natural inclusion $X_i \to X$, $x \mapsto (x, i)$, and X is the disjoint union of the images. We equip X with the direct sum topology. This means that the (images of the) X_i are open subsets and that the induced topology is the original one. Then one defines in an obvious way a Radon measure on X such that the restriction to the X_i are the given dx_i. □

Proof of Proposition 3.3. To any bounded continuous function $h : \mathbb{R} \to \mathbb{C}$ we associate the functional

$$I_h : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}, \quad I_h(g) = \int_{-\infty}^{\infty} h(t)\hat{g}(t)dt.$$

The integral exists, since $\hat{g}$ and hence hg are rapidly decreasing. We apply this to the function $h(t) = \langle U(t)a, a\rangle$ where a is a cyclic vector. This function has the property

$$h(-t) = \langle U(-t)a, a\rangle = \langle a, U(t)a\rangle = \overline{h(t)}.$$

Using this it is easy to check that I_h is real, i.e. real valued for real h. We will see a little that I_h is actually a Radon measure on $\mathcal{C}_c^\infty(\mathbb{R})$. For this reason, we later use already now the notation

$$\int_{\mathbb{R}} g(x)d\mu = \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \hat{g}(t)dt \qquad (g \in \mathcal{C}_c^\infty(\mathbb{R})).$$

Next we define a linear map

$$W : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow H, \quad g \longmapsto \int_{-\infty}^{\infty} \hat{g}(t)U(t)adt.$$

This is a Bochner integral with values in the Hilbert space H. The integrand is continuous, hence measurable and it is bounded by the integrable function $|\hat{g}|$. Hence the Bochner integral exists.

For $g_1, g_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ we compute

$$\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu$$

as follows. We make use of the fact that the Fourier transformation of the product $g_1 g_2$ of two functions equals the convolution of the two Fourier transforms

$$\widehat{g_1 g_2} = \hat{g}_1 * \hat{g}_2.$$

Recall that the convolution of two functions on the line is

$$(g_1 * g_2)(x) = \int_{-\infty}^{\infty} g_1(x-t)g_2(t)dt.$$

So we get

$$\begin{aligned}\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu &= \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \widehat{g_1\bar{g}_2}(t)dt \\ &= \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \int_{-\infty}^{\infty} \hat{g}_1(t-s)\hat{\bar{g}}_2(s)ds.\end{aligned}$$

We compare this with the inner product of $W(g_1)$ and $W(g_2)$ in the Hilbert space H. It is

$$\langle W(g_1), W(g_2)\rangle = \left\langle \int_{-\infty}^{\infty} \langle U(t)a, a)\rangle \hat{g}_1(t)dt, \int_{-\infty}^{\infty} \langle U(s)a, a)\rangle \hat{g}_2(s)dtds \right\rangle.$$

The integrals are standard integrals along continuous functions with compact support. They can be considered as Riemann integals and hence approximated by finite sums. In this way we see

$$\langle W(g_1), W(g_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle U(t)a, a)\rangle \overline{\langle U(s)a, a\rangle} \hat{g}_1(t)\hat{\bar{g}}_2(s)dtds.$$

By means of the integral transformation $(t,s) \mapsto (t-s,s)$ we obtain

$$\boxed{\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu = \langle W(g_1), W(g_2)\rangle}$$

Now, let $g \in \mathcal{C}_c^\infty(\mathbb{R})$ be a real nonnegative function. In the case that $\sqrt{g}$ is differentiable, we set $g_1 = g_2 = \sqrt{g}$ to show that $\int g d\mu$ is non negative. But $\sqrt{g}$ needs not to be differentiable (notice that the square root of x^2 is $|x|$ which is not differentiable at the origin). But it is always possible to approximate g by functions g_1^2 where g_1 is differentiable So we see that $d\mu$ is a Radon measure as we have claimed. The map $W : \mathcal{C}_c^\infty(\mathbb{R}) \to H$ is unitary. Hence it is injective and it extends to a unitary map

$$L^2(\mathbb{R}, d\mu) \longrightarrow H.$$

(Here one uses that y bounded linear map $E \to F$ of normed spaces extends to the completions.) In particular, W extends to $S(\mathbb{R})$. The image is a complete and hence a closed subspace of H.

Next we prove $W\tilde{U}(s) = U(s)W$. This follows simply from the fact that the Fourier transform of the function $x \mapsto e^{\mathrm{i}sx}g(x)$ is the function $x \mapsto \hat{g}(x-s)$.

It remains to show that W is surjective. Here we have to use that a is a cyclic vector. It is sufficient that a is in the image, or even that there exists a sequence g_n of tempered functions such that $W(g_n) \to a$. For this purpose we choose a differentiable Dirac sequence h_n. Then the integrals $\int h_n(t)U(t)a$ converge to $U(0)a = a$. We can write $h_n = \hat{g}_n$ where g_n is tempered. □

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