Unitary Representations

Contents

Chapter I. Representations

1. Generalities about Banach- and Hilbert spaces

Usually, we consider only vector spaces over the field of complex numbers if not stated otherwise. A norm on a vector space E is a real valued function $\|\cdot\|$ on E with the properties $\|a\| \geq 0$ and $= 0$ only for $a = 0$, $\|Ca\| = |C|\|a\|$, $\|a + b\| \leq \|a\| + \|b\|$ $(a, b \in E)$, $C \in \mathbb{C})$. Then $\|a - b\|$ is a metric on E. The normed space E is called complete, or a Banach space, if every Cauchy sequence converges. Every normed space E can be embedded into a Banach space $\bar{E}$ as a dense subspace (with the restricted norm) in an essentially unique manner. One calls $\bar{E}$ the completion of E. Let $F \subset E$ be a linear subspace of a Banach space. It is a closed subspace if and only if it is a Banach space (with respect to the restricted norm). The closure of a linear subspace in a Banach space is a linear subspace and hence a Banach space. Since any two norms on a finite dimensional vector space are equivalent, every finite dimensional normed vector space is a Banach space. As a consequence, every finite dimensional subspace of a normed vector space is closed.

A linear map $A : E \to F$ between normed vector spaces is called *bounded* if there exists a constant $C \geq 0$ such that $\|Aa\| \leq C\|a\|$ for all $a \in E$. Then there exists a smallest number C with this property. It is called the norm of A and is denoted by $\|A\|$. We mention that A is bounded if and only if it is continuous (at the origin is enough). For finite dimensional E each linear map is bounded. We denote be $\mathrm{Hom}(E, F)$ the space of all bounded linear maps. If F is a Banach space, this is a Banach space as well. In the case $E = F$ we write $\mathrm{Hom}(E, E) = \mathrm{End}(E)$. If A is bounded and bijective, then A^{-1} is bounded as well. The group of all invertible bounded operators in $\mathrm{End}(E, E)$ is denoted by $\mathrm{GL}(E)$.

We will make use of the *theorem of uniform boundedness*:

1.1 Theorem. *Let E be a Banach space and let $\mathcal{M} \subset \mathrm{End}(E)$ be a set of bounded operators such that $\{Aa,\ a \in E\}$ is bounded for all $a \in E$. Then $\mathcal{M}$ is a bounded subset of* $\mathrm{End}(E)$.

All what we have said do far about complex Banach spaces can be formulated and is true for real Banach spaces.

A hermitian form on a vector space E is a function $\langle\cdot,\cdot\rangle : E\times E\to\mathbb{C}$ which is linear in the first variable and which has the property $\langle a,b\rangle=\overline{\langle b,a\rangle}$. It is called positive definite if $\langle a,a\rangle>0$ for all $a\neq 0$. Then $\|a\|:=\sqrt{\langle a,a\rangle}$ is norm. We call $(E,\langle\cdot,\cdot\rangle)$ a Hilbert space if it is a Banach space with this norm.

We will make use of the Theorem of Riesz:

Let $L : H\to\mathbb{C}$ be a continuous linear functional on a Hilbert space H. Then there exists a unique vector $a\in H$ such that $L(x)=\langle x,a\rangle$ (and each linear functional of this kind is continuous and has the norm $\|L\|=\|a\|$).

These special linear forms show that for every vector $a\in H$, $a\neq 0$, there exists a continuous linear functional L with the property $L(a)\neq 0$.

This statement is also true for Banach spaces. From the theorem of Hahn-Banach follows the following result:

For each non-zero vector $a\in E$ of a Banach space there exists a continuous linear functional L with the property $L(a)\neq 0$.

We will make use of another important result about Hilbert spaces. Let $A\subset H$ be a closed linear subspace. Denote by

$$B=\{b\in H;\ \langle a,b\rangle=0 \text{ for all } a\in A\}$$

the orthogonal complement of A. This is a closed linear subspace and one has $H=A\oplus B$.

A family $(a_i)_{i\in I}$ is called an orthonormal system if any two members with different indices are orthogonal and if the norm of each member is one. A *Hilbert space basis* is by definition a maximal orthonormal system. It is easy to show (using Zorn's lemma and the above remark about orthogonal complements) that Hilbert space bases exist. Even more, every orthonormal system is contained in a maximal one.

A Hilbert space H is called separable if it contains a countable dense subset. One can show that this is the case if and only if each Hilbert space basis is finite or countable. We give an example. The space ℓ^2 consists of all sequences $(a_1,a_2,\dots)$ of complex numbers such that $\sum|a_n|^2$ converges. It can be shown that for two $a,b\in\ell^2$ the series

$$\langle a,b\rangle=\sum a_n\bar b_n$$

converges absolutely and equips ℓ^2 with the structure as a Hilbert space. The usual unit vectors (1 at one place and 0 at the others) give a Hilbert space basis. Let now H be any infinite dimensional separable Hilbert space with a Hilbert space basis $e_1,e_2,\dots$. For each $a\in\ell^2$ the series

$$\sum_{n=1}^\infty a_ne_n:=\lim_{N\to\infty}\sum_{n=1}^N a_ne_n$$

then converges in H. We mention that a series $a_1 + a_2 + \cdots$ in a Banach space converges if it converges absolutely in the sense that $\|a_1\| + \|a_2\| + \cdots$ converges. This gives a map

$$\ell^2 \xrightarrow{\sim} H.$$

This map is actually an isomorphism of Hilbert spaces (which means that it is an isomorphism of vector spaces which preserves the Hermitian forms). Hence all infinite dimensional separable Hilbert spaces are isomorphic as Hilbert spaces. (The same kind of argument shows a standard result of linear algebra, namely that two finite dimensional Hibert spaces are isomorphic as Hilbert spaces if and only if their dimensions agree.)

Assume that $H_1,\ H_2, \ldots$ is a sequence of pairwise orthogonal closed subspaces of the Hilbert space H. Assume that their algebraic sum is dense in H. If we choose a Hilbert space basis in each H_i and collect them, we get a Hilbert space basis of H. This shows that every $a \in H$ has a unique representation as absolute convergent series $a = a_1 + a_2 + \cdots$ where $a_i \in H_i$. We write this as

$$H = \widehat{\bigoplus_i} H_i$$

and call this a direct Hilbert sum.

There is an abstract version of this. Let H_n be a family of Hilbert spaces. We define H to be the set of all sequences (h_n), $h_n \in H_n$ such that $\sum \|h_n\|$ converges. There is a natural imbedding of H_n into H. The image $\tilde{H}_n$ consists of all elements of H such only the nth component can be different from 0. The space H carries a natural structure as Hilbert space and it is the direct Hilbert of the $\tilde{H}_n$. Usually one identifies $\tilde{H}_n$ with H_n and calls H the direct Hilbert sum of the H_n.

2. Generalities about measure theory

All topological spaces that carry measures are assumed to be Hausdorff and to have a countable basis of the topology. The latter means that there exists a countable system of open subsets such that each open subset can be written as a union of sets from this system. Every metric space with an countable dense subset (for example $\mathbb{C}^n$) has this property. Every subspace (equipped with the induced topology) keeps this property.

We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on a locally compact space X and by $\mathcal{C}_c(X)$ the subset of all continuous functions with compact support. A Radon measure is a linear functional $I : \mathcal{C}_c(X) \to \mathbb{R}$ which is real in the sense $I(\bar{f}) = \overline{I(f)}$ and positive in the sense that $I(f) \geq 0$

for real $f \geq 0$. Usually one writes

$$I(f) = \int_X f(x)dx.$$

We assume that the reader is familiar with some way to extend a Radon measure to the class of integrable functions. We just indicate the steps, how this can be done.

One introduces $\mathbb{R} \cup \{\infty\}$ as ordered set ($x \leq \infty$ for all x). Every non-empty set $M \subset \mathbb{R} \cup \{\infty\}$ has a smallest upper bound $\operatorname{Sup}(M)$ in $\mathbb{R} \cup \{\infty\}$. One extends the addition to $\mathbb{R} \cup \{\infty\}$ by $x + \infty = \infty + x$ for all x and similarly the mutiplication with a positive $C > 0$ by $C\infty = \infty$.

A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called a *Baire function* if there exists an increasing sequence $f_n \in \mathcal{C}_c(X)$, $f_1 \leq f_2 \leq \ldots$ such that $f(x) = \operatorname{Sup}\{f_n(x);\ x \in X\}$. One can show that $\operatorname{Sup}\{I(f_n)\}$ is independent of the choice of the sequence. Every $f \in \mathcal{C}_c(X)$ is a Baire function and in this case $\operatorname{Sup}\{f_n(x);\ x \in X\}$ agrees with $I(f)$. Hence we can define $I(f) = \operatorname{Sup}\{I(f_n)\}$ for arbitrary Baire functions. We mention that the function "constant ∞" is a Baire function. Hence we can define for an arbitrary nowhere negative function $f : X \to \mathbb{R} \cup \{\infty\}$

$$\bar{I}(f) = \operatorname{Inf}\{I(h); \quad f \leq h \text{ Baire function}\}.$$

The general rule $\bar{I}(f + g) \leq \bar{I}(f) + \bar{I}(g)$ holds.

Now one can define integrable functions:

A function $f : X \to \mathbb{R}$ is called integrable if there there exists s sequence $f_n \in \mathcal{C}_c(X)$ such that $\bar{I}(|f - f_n|)$ is finite and tends to zero.

One can show that then $I(f_n)$ converges. One can show even more that Baire functions f with finite $I(f)$ (for example elements of $\mathcal{C}_c(X)$) are integrable and that $I(f) = \lim I(f_n)$ in this case. Hence we can define $I(f) = \bar{I}(f)$ for $f \in \mathcal{L}^1(X, dx)$. With this definition it is easy to see that the space $\mathcal{L}^1(X, dx)$ of all integrable functions is a vector space. It has the property that with f also $|f|$ is integrable. The integral is a linear functional on $\mathcal{L}^1(X, dx)$ with the property $I(f) \geq 0$ for $f \geq 0$.

A function $f : X \to \mathbb{R}$ is called a *zero function* if $\bar{I}(|f|) = 0$. This means that for each $\varepsilon > 0$ there exists a Baire function h with $|f| \leq h$ and $I(h) < \varepsilon$. A subset of X is called a zero subset if its characteristic function is a zero function. Zero functions are integrable. If f is integrable and g is a function that coincides with f outside a zero set then g is integrable too and $I(f) = I(g)$.

We recall the basic limit theorems:

2.1 Theorem of Beppo Levi. *Assume that $f_1 \leq f_2 \ldots$ is an increasing sequence of integrable functions such that the sequence of their integrals is bounded. Then the pointwise limit $f(x) = \lim f_n(x)$ exists outside a zero set.*

If one defines $f(x)$ arbitrarily for this zero set, one gets an integrable function with the property

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

2.2 Lebesgue's limit theorem. *Let $f_n(x)$ be a pointwise convergent sequence of integrable functions. Assume that there exists an integrable function h with the property $|f_n(x)| \le h(x)$ for all n and x. Then $f(x) = \lim f_n(x)$ is integrable and one has*

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

The subset $\mathcal{N} \subset \mathcal{L}^1(X, dx)$ of zero functions is a sub-vector space and the integral factors through the quotient

$$L^1(X, dx) := \mathcal{L}^1(X, dx)/\mathcal{N}.$$

From the limit theorems one can deduce that this space gets a Banach space with the norm

$$\|f\|_1 := \sqrt{\int_X |f(x)|dx}.$$

(Usually we will denote the class of an element $f \in L^1(X, dx)$ in $\mathcal{L}(X, dx)$ by the same letter f. A more careful notation would be to use a notation like $[f]$ for the class. For sake of simplicity we avoid this as long it is clear whether we talk about f or of its class.)

Let us assume that the Radon measure is non-trivial in the following sense: Let $f \in \mathcal{C}_c(X)$ be a non-negative function with the property $I(f) = 0$. Then $f = 0$. For such a measure the natural map

$$\mathcal{C}_c(X) \longrightarrow L^1(X, dx)$$

is injective and $L^1(X, dx)$ is the completion of $\mathcal{C}(X)$ with respect to the norm $\|\cdot\|_1$. Hence integration theory can be understood as a concrete realization of the completion.

There is another important notion:

A function $f : X \to \mathbb{C}$ is called ***measurable*** *if for any non-negative function $h \in \mathcal{C}_c(X)$ the function*

$$f_h(x) := \begin{cases} f(x) & \text{if } -h(x) \le f(x) \le h(x), \\ 0 & \text{else} \end{cases}$$

is integrable.

Integrable functions are measurable. All continuous functions are measurable. Measurability is conserved under all kind of standard constructions of functions which are used in analysis as addition and multiplication of functions but also taking pointwise limits and constructions as sup, inf, lim sup, lim inf for sequences of functions. So the statement *"all functions are measurable"* is not really true but nearly true. (Counter examples need sophisticated application of the axiom of choice.)

A function f is integrable if and only if it is measurable and if $\bar{I}(|f|) < \infty$. Together with the previous remark this means that integrability means a kind of boundedness.

Let $p \geq 1$. The spaces $\mathcal{L}^p(X, dx)$ consist of all measurable functions f such that $|f|^p$ is integrable. This is the case for zero functions. One defines

$$\|f\|_p := \int_X \sqrt[p]{\int_X |f(x)|^p dx}.$$

This satisfies the triangle inequality. It induces a norm on the space

$$L^p(X, dx) = \mathcal{L}^p(X, dx)/\mathcal{N}$$

which is a Banach space with this norm. The case $p = 2$ is of special importance. One can consider on $\mathcal{L}^2(X, dx)$ the hermitian form

$$\langle f, g\rangle := \int_X f(x)\overline{g(x)}dx.$$

This induces a positive definite form on $L^2(X, dx)$ and equips this space with a structure as separable Hilbert space.

As a special example one can take the space $X = \mathbb{N}$ equipped with the discrete topology and the Radon measure $I(a) = \sum_n a_n$. The associated L^2-space is ℓ^2.

There is an extension of measure theory, the Bochner integral. For a Banach space E we can consider the space of compactly supported continuous functions $\mathcal{C}_c(X, E)$ with values in E.

2.3 Lemma. *Let (X, dx) be a Radon measure and E a Banach space. There exists a unique linear map*

$$\mathcal{C}_c(C, E) \longrightarrow E, \quad f \longmapsto \int_X f(x)dx,$$

such that for each continuous linear functional $L : E \to \mathbb{C}$ one has

$$L\Big(\int_X f(x)dx\Big) = \int_X L(f(x))dx.$$

The uniqueness follows directly from the Hahn-Banach theorem. So the existence, but not so quite obvious. Since for our purposes it would be sufficient to treat the case of Hilbert spaces we mention that the existence in this case is a direct consequence of the Theorem of Riesz.

There is also the notion of a measurable function. We only need it in the case where E is separable which means that it contains a countable dense subset. Then a function $f : X \to E$ is measurable if and only if its composition with all continuous linear forms is measurable. A measurable function $f : X \to E$ is called a zero function if it is zero outside a zero set. Now the spaces $\mathcal{L}^p(X, E, dx)$ can be defined in the same way as in the case $E = \mathbb{C}$. They contain the space $\mathcal{N}$ of zero functions and the quotients $L^p(X, E, dx)$ are Banach spaces. If $E = H$ is a Hilbert space, the space $L^2(X, H, dx)$ gets a Hilbert space with an obvious inner product.

3. Generalities about Haar measures

A topological group G is a group which carries also a topology such that the maps

$$G \times G \longrightarrow G,\ (g, h) \longmapsto gh, \quad G \longrightarrow G,\ g \longmapsto g^{-1},$$

are continuous. Here $G \times G$ has been equipped with the product topology. A *locally compact group* is a topological space whose underlying space is locally compact. We always assume that G has a countable basis of the topology.

A Haar measure on a locally compact group G is a non-zero left invariant Radon measure

$$\int_G f(x)dx = \int_G f(gx)dx \qquad (g \in G).$$

We make use of the fact that a non zero Haar measure always exists and is uniquely determined up to a constant factor.

The usual integral on $\mathbb{R}$ is a Haar measure on the additive group $\mathbb{R}$ and a Haar measure on the multiplicative group $\mathbb{R}^\bullet$ is given by

$$\int_{\mathbb{R}^\bullet} f(t)\frac{dt}{t}$$

where dt is the usual measure.

If $f \in \mathcal{C}_c(G)$ is a function with the properties $f \geq 0$ and $I(f) = 0$. Then $f = 0$. Hence we have $\mathcal{C}_c(G) \hookrightarrow L^p(G, dx)$.

Let $g \in G$. Then

$$f \longmapsto \int_G f(xg)dx$$

is also left invariant. Hence there exists a positive real number $\Delta(g) = \Delta_G(g)$ with the property

$$\int_G f(xg^{-1})dx = \Delta(g) \int_G f(x)dx.$$

The function $\Delta : G \to \mathbb{R}_{>0}$ is of course independent of the choice of dx. It is called the *modular function* of G. It is clearly a continuous homomorphism, $\Delta(gh) = \Delta(g)\Delta(h)$.

3.1 Lemma. *For every function $f \in \mathcal{L}^1(G, dx)$ the formula*

$$\int_G f(x^{-1})\Delta(x^{-1})dx = \int_G f(x)dx$$

holds.

Proof. One can check that the integral on the left hand side is a Haar measure. Hence it agrees with the right hand side up to constant a factor $C > 0$. Applying the formula twice we get $C^2 = 1$ and hence $C = 1$. □

The group G is called *unimodular* if $\Delta(g) = 1$ for all g. There are three obvious classes of unimodular groups:

1) Abelian groups are unimodular.
2) A group G is unimodular if its commutator subgroup is dense.
3) Compact groups are unimodular, more generally, for arbitrary G the restriction of Δ_G to any compact subgroup is trivial.

The last statement is true since the only compact subgroup of the multiplicative group of positive reals is $\{1\}$.

We give an example of a group which is not unimodular. Let $P \subset \mathrm{SL}(2, \mathbb{R})$ be the group of all upper triangular matrices of determinant 1. Each p can be written in the form

$$p = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad (a \neq 0).$$

Moreover the map

$$\mathbb{R}^\bullet \times \mathbb{R} \xrightarrow{\sim} P, \quad (a, n) \longmapsto p,$$

is topological. Hence we can identify $\mathcal{C}_0(P)$ and $\mathcal{C}_c(\mathbb{R}^\bullet \times \mathbb{R})$.

3.2 Lemma. *Let $P \subset \mathrm{SL}(2, \mathbb{R})$ be the group of upper triangular matrices. Let da be a Haar measure on $\mathbb{R}^\bullet$ and dn a Haar measure on $\mathbb{R}$. Then then measure*

$$\int_P f(p) := \int_{\mathbb{R}^\bullet} \int_{\mathbb{R}} f(an)da\,dn$$

is a Haar measure. The modular function is

$$\Delta(p) = a^2.$$

(One can also write $\int\int f(an)dndn$ for the right hand side, since orders of integration can be interchanged, but $\int\int f(na)dadn$ would be false.)

Proof. The proof can be given by a simple calculation which rests on the formula

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ 0 & a^{-1} \end{pmatrix}. \qquad \square$$

We also need quotient measures. Let $H \subset G$ be a closed subgroup of a locally compact group G. Then H is also locally compact. We consider the coset space $H\backslash G$ that consists of all right cosets Hg. This is the quotient space of G by the natural action of H (multiplication from the right.) We equip it with the quotient topology with respect to the natural projection $G \to H\backslash G$. Then this projection is continuous and open. We claim that $H\backslash G$ is Hausdorff. Hausdorff means that the diagonal in $H\backslash G \times H\backslash G$ is closed. This means that its inverse image in $G \times G$ is closed. But this inverse image of H with respect to the map $G \times G \to G$, $(x, y) \mapsto xy^{-1}$.

Since $G \to H\backslash G$ is open, the space $H\backslash G$ is locally compact. There is also a natural continuous map

$$(H\backslash G) \times G \longrightarrow H\backslash G, \quad (Hg_1, g_2) \longmapsto Hg_1g_2$$

which as action from the right. A Radon measure dx on $X = H\backslash G$ is called G-invariant if

$$\int_{H\backslash G} f(xg)dx = \int_{H\backslash G} f(x)dx.$$

3.3 Proposition. *Let $H \subset G$ be a closed subgroup. Let dx be a Haar measure on G and dh a Haar measure on H Assume that $\Delta_G|H = \Delta_H$. Then there exists a non-zero invariant Radon measure dy on $H\backslash G$ and this Radon measure is unique up to a positive constant factor. It has the following property. Let dh be a Haar measure on H. Then*

$$\int_G f(x)dx = \int_{H\backslash G}\left[\int_H f(hy)dh\right] dy$$

is a Haar measure on G.

We should mention that the function $y \mapsto \int_H f(yh)dh$ can be considered as a function on $H\backslash G$. It is continuous and with compact support there.

We indicate the general proof of the existence of an invariant measure. A function $h : H\backslash G \to \mathbb{C}$ is called special if it can be written in the form $h(y) = \int_H f(hy)dh$ with a function $f \in \mathcal{C}_c(G)$. It is not difficult to show that there are many special functions such that is sufficient to define a radon measure on them. For the special h we define the integral $\int_{H\backslash G} f(y)dy$ using the formula in Proposition 3.3 where on the left hand side a Haar measure is taken. There is a problem. The function h does not determine f uniquely. Hence one has to prove a Lemma. One has to show

$$\int_H f(hy)dh = 0 \Longrightarrow \int_G f(x)dx.$$

It is a good exercise to to this for a finite group G. The integral then just are finite sums. In the general case the condition $\Delta_G|H = \Delta_H$ will play a role. In this way we get the existence of an invariant measure on $H\backslash G$ such the claimed formula holds. The proof of the uniqueness of the quotient measure is the same as the proof of the uniqueness of the Haar measure. □

We also mention that the formula in Proposition 3.3 holds for all $f \in \mathcal{L}^1(G, dx)$ with the usual caution: the inner integral exists outside of a set of measure zero and gives – extended arbitrarily – an integrable function on $H\backslash G$.

4. Generalities about representations

A representation π of a group G on a complex vector space is a homomorphism $\pi : G \to \mathrm{GL}(V)$ of G into the group of $\mathbb{C}$-linear automorphisms of V. Frequently we will write $g(a)$ or even simply ga instead of $\pi(g)(a)$. The map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

then has the properties:

1) $ea = a$ for all $a \in V$ (e denotes the unit element of G).
2) $(gh)a = g(ha)$ for all $g, h \in G$, $a \in V$.
3) $g(a + b) = g(a) + g(b)$, $g(Ca) = Cga$ $(C \in \mathbb{C})$.

Conversely, a map with the properties 1)-3) comes from a unique representation π.

Left and Right

Let G be a group and V simply a set. A map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

with the properties 1)-2) is also called an action of G *from the left* on V. If one replaces in 2) the condition by $(gh)a = h(g(a))$ one gets the notion of an action *from the right.* This looks better if one uses the notation ag instead of ga since then the rule takes the better looking form $a(gh) = (ag)h$. If ga is an action from the left then $g^{-1}a$ is an action from the right, and conversely.

Hence there is no essential difference between the two. Keep in mind that due to our definition representations are actions from the left.

Let G be a locally compact group and E a Banach space. A representation π of G on E is called continuous if the map

$$G \longrightarrow E, \quad g \longmapsto \pi(g)(a),$$

is continuous for all $a \in E$. From the uniform boundedness theorem we can conclude:

4.1 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a continuous representation and $K \subset G$ a compact subset. Then the set $\pi(K)$ is bounded in* $\mathrm{End}(E)$.

The condition of continuity in the definition of a representation can be weakened.

4.2 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a homomorphism with the following two properties:*

1) *There is a neighborhood of the identity whose image in* $\mathrm{GL}(E)$ *is bounded.*
2) *There is a dense subset of vectors $a \in E$ such that $g \mapsto \pi(g)(a)$ is continuous.*

Then π is a continuous representation.

Proof. We have to show that for fixed a the function $x \mapsto \pi(x)a$ is continuous. It is obviously enough to proof this at the unit element $x = e$. Hence we have to estimate $\|\pi(x)a - a\|$. For some b in the dense subset we use the estimate

$$\|\pi(x)a - a\| \le \|\pi(x)a - \pi(x)b\| + \|\pi(x)b - b\|\|b - a\|.$$

If we choose b close enough to a we obtain the desired result. □

Algebraic Irreducibility

Let $\pi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \subset V$ is called invariant if $g \in G$ and $a \in W$ implies $ga \in W$. Then we obtain a representation $\pi' : G \to \mathrm{GL}(W)$. A representation $\pi : G \to \mathrm{GL}(V)$ is called *algebraically irreducible* if $V \ne 0$ and if besides $\{0\}$ and V there are no invariant subspaces. Let W_1, W_2 be two invariant subspaces of V. Then $W_1 + W_2$ and $W_1 \cap W_2$ are also invariant. If W_1 and W_2 are irreducible then either they are equal or their intersection is zero.

Topological Irreducibility

Let now $\pi : G \to \mathrm{GL}(V)$ be a *continuous* representation. It is called *topologically irreducible* if there is no *closed* invariant subspace different from $\{0\}$ and V.

For finite dimensional representations (this means that V is finite dimensional) algebraic and topological irreducibility is the same.

A representation of a topological group on a Hilbert space H is called *unitary* if it is continuous and if all operators $\pi(g)$ are unitary operators. This means concretely

$$\langle ga, gb \rangle = \langle a, b \rangle$$

for $a, b \in H$ and $g \in G$. It is enough to demand this for $a = b$. If we talk about an irreducible unitary representation, we always mean that it is topologically irreducible.

We describe a fundamental example of a unitary representation. Let G be a locally compact group. We consider a closed subgroup $H \subset G$. For sake of simplicity we assume that both are unimodular. Then dx is left- and right invariant. We consider the space of right cosets $H\backslash G$. The group G acts on $H\backslash G$ by multiplication from the right. This is an action from the right. Let $f : H\backslash G \to \mathbb{C}$ be a function and $g \in G$. We define the translate $R_g f$ of f by $(R_g f)(x) = f(xg)$. This is an action from the left of G on the set of function on $H\backslash G$. This defines a map

$$R : G \longrightarrow \mathrm{GL}(L^2(H\backslash G, dx)).$$

By means of 1.1 one can show that this representation is continuous. It is obviously a unitary representation. In the special case $H = \{e\}$ one obtains the so-called regular representation of G on $L^2(G)$.

One of the basic problems of harmonic analysis is the investigation of this representation and to describe its spectral decomposition. This problem has been studied for the regular representation of semi simple groups G (for example $\mathrm{SL}(n, \mathbb{R})$) by Harish Chandra. In the theory of automorphic forms one studies the case where $H = \Gamma$ is a discrete subgroup such that $\Gamma\backslash G$ has finite volume.

What means “spectral decomposition”? This is not so easy to explain and not the goal of these notes. Nevertheless it is useful to get an idea of it. We give two examples. The first example is the group S^1 of complex numbers of absolute value one (circle group). The functions f on S^1 correspond to the periodic functions (period 2π) F on $\mathbb{R}$ through

$$F(t) = f(\exp(2\pi \mathrm{i} t)).$$

From the theory of Fourier series one knows that $L^2(S^1)$ is the direct Hilbert sum of the one dimensional subspaces $H(n)$ spanned by $f(\zeta) = \zeta^n$ $(n \in \mathbb{Z})$.

These are invariant subspaces. The spectral decomposition of the regular representation of S^1 is

$$L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} H(n).$$

The second example deals with the regular representation of $\mathbb{R}$. There are also one dimensional spaces $H(t)$ generated by the function $x \mapsto e^{2\pi i t x}$ which are invariant under translations $t \mapsto t + a$. Now t can be an arbitrary real number. But the difference is that now $H(t)$ is not contained in $L^2(\mathbb{R})$. Nevertheless the theory of Fourier transform shows that all f in a certain dense subspace of $L^2(\mathbb{R})$ can be written in a unique way in the form

$$f(t) = \int_{-\infty}^{\infty} g(t) e^{2\pi i t} dt.$$

Hence one is tempted to say that $L^2(\mathbb{R})$ is the direct integral of the spaces $H(t)$ and to write this in the form

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} H(t) dt.$$

For general G the spectral decomposition will include both types (discrete and continuous spectra) and the constituents will not be one-dimensional but irreducible unitary representations (often infinite dimensional).

Intertwining Operators

A morphism between two continuous representations $\pi_i : G \to \mathrm{GL}(E_i)$ on Banach spaces is a continuous linear map $E_1 \to E_2$ which is compatible with the action of G in an obvious sense. Such morphisms are also called "intertwining operators". It is clear what it means that an intertwining operator is an isomorphism. If $F \subset E$ is a closed G-invariant subspace then the natural inclusion $F \hookrightarrow E$ is a morphism. We call (G, F) a sub-representation of (G, E).

For unitary representations we will make use of a more restrictive notion of isomorphy. An isomorphism $H_1 \to H_2$ between two unitary representations $\pi : G \to \mathrm{GL}(H_i)$ is called a *unitary isomorphism,* or an isomorphism of unitary representations, if the isomorphism $H_1 \to H_2$ is an isomorphism of Hilbert spaces. This means that it preserves the scalar products.

5. The convolution algebra

let G be a locally compact group with a chosen Haar measure. The convolution of two functions $f, g \in \mathcal{C}_c(G)$ is defined by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy.$$

The convolution defines an associative product on $\mathcal{C}_c(G)$. We leave the proof of the associativity as an exercise. Hence $\mathcal{C}_c(G)$ has the structure of an associative $\mathbb{C}$-algebra.

Let $\pi : G \to \mathrm{GL}(H)$ be a continuous representation on a Banach space. For any $f \in \mathcal{C}_c(G)$ and any $h \in H$ we can consider the function

$$G \longrightarrow H, \quad x \longmapsto f(x)\pi(x)h.$$

It is continuous and with compact support. Hence we can define the integral

$$\int_G f(x)\pi(x)h dx.$$

If we vary h we get an operator $H \to H$. One can check that it is linear and continuous.

We denote this operator by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

One verifies

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2).$$

What we obtain is an algebra homomorphism

$$\pi : \mathcal{C}_c(G) \longrightarrow \mathrm{End}(H).$$

The image of π consists of continuous linear operators $T : H \to H$.

Now we assume that H is a Hilbert space. We denote the adjoint of an operator $T \in \mathrm{End}(H)$ by T^*. It is defined by the formula $\langle Tx, y\rangle = \langle x, T^*y\rangle$. The existence of T follows from the Riesz lemma. Of course T^* is continuous as T, and moreover both have the same norm.

We define

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}.$$

We now assume that π is unitary. It is easy to check in this case that the new map π has the property that

$$\pi(f^*) = \pi(f)^*.$$

What we obtained is a $*$-algebra representation. We describe briefly what this means. An (associative) algebra A is a vector space (in our case over $\mathbb{C}$) together with a bilinear map

$$A \times A \longrightarrow A, \quad (a, b) \longmapsto ab.$$

We assume that this is associative but we do not assume that A contains a unit element. An involution on A is a map

$$A \longrightarrow A, \quad a \longmapsto a^*,$$

with the properties

a) $(a + b)^* = a^* + b^*, \quad (Ca)^* = \bar{C}a^*,$
b) $(ab)^* = b^*a^*.$
c) $a^{**} = a.$

5.1 Definition. *A $*$-algebra $(A, *)$ is an associative algebra (not necessarily with unit) together with a distinguished involution*

An example of a $*$-algebra is the convolution algebra $\mathcal{C}_c(G)$ with the involution defined above. Another example of a $*$-algebra is the space $\mathrm{End}(H)$ of continuous linear operators on a Hilbert space H. Multiplication is the composition of operators and the $*$-operator is given by the adjoint.

By a representation of an algebra A on a vector space V one understands a linear map $A \to \mathrm{End}(V)$ which is compatible with multiplication. By a $*$-algebra representation of a $*$-algebra A on a Hilbert space H we understand a representation

$$A \longrightarrow \mathrm{End}(H)$$

such the image of A consists of continuous operators and that is also compatible with the star operators. What we have seen that a unitary representation $\pi : G \to \mathrm{GL}(H)$ induces a $*$-algebra representation $\pi : \mathcal{C}_c(G) \to \mathrm{End}(H)$.

There are obvious notions of irreducibility:

A representation $A \to \mathrm{End}(V)$ of an algebra is called algebraically irreducible if the image of A is not zero and if there is no invariant subspace of V different form 0 and V.

A $*$-algebra representation $A \to \mathrm{End}(H)$ is called topologically irreducible if the image if A is non zero and if there is no closed invariant subspace of H different from 0 and H.

An example of a finite dimensional algebra representation is the tautological representation of $A = \mathrm{End}(V)$ on V. It is just the identity map $\mathrm{End}(V) \to \mathrm{End}(V)$. At least in the finite dimensional case it is clear that this representation is irreducible. A special case of a fundamental structure theorem of Wedderburn states (in the case of the ground field $\mathbb{C}$):

5.2 Theorem. *Let $\pi : A \to \mathrm{End}(V)$ be an irreducible representation of an algebra A on a finite dimensional vector space V. Then π is surjective.*

We don't give the proof here and refer to the text book of S. Lang on algebra. To be honest, we mention that Lang treats only the case where A contains a unit element. The general case can be reduced by the technique of adjoining a unit element.

A trivial consequence of Theorem 5.2 is as follows. Let $T : V \to V$ be a linear operator that commutes with all $\pi(a)$, $a \in A$. Then T is a multiple of the identity. A basic result states that this carries over to the infinite dimensional case.

5.3 Theorem (Schur's lemma for algebra representations). *Let π be a topologically irreducible unitary representation of a $*$-algebra A on a Hilbert space H. Assume that $T : H \to H$ is a linear and continuous operator that commutes with all $\pi(a)$, $a \in A$. Then T is a constant multiple of the identity.*

Corollary. *If A is abelian then then H is one-dimensional.*

We will not give the proof in the infinite dimensional case in this text. We just mention that it rests on the spectral theorem for normal operators. □

The same theorem is true for irreducible unitary representations of locally compact groups. Actually it is a consequence of Theorem 5.3 as we shall point out. The argument would be very easy if there exists for $g \in G$ a Dirac function $\delta_g \in \mathcal{C}_c(G)$ which means

$$\delta_g(x) = 0 \text{ for } x \neq g \quad \text{and} \quad \int_G \delta_g(x)dx = 1.$$

Such a situation is of course rare, but it occurs, namely for finite groups. A simple computation then gives $\pi(\delta_g) = \pi(g)$. From this one can deduce that a subspace of H is invariant under all $\pi(g)$, $g \in G$, if and only if it is invariant under all $\pi(f)$, $f \in \mathcal{C}_c(G)$. Actually there is a weak variant of Dirac functions.

5.4 Lemma. *For each locally compact group G there exists a sequence of functions $\delta_n \in \mathcal{C}_c(G)$ with the following properties.*

1) $\mathrm{supp}(\delta_{n+1}) \subset \mathrm{supp}(\delta_n)$.
2) *For each neighborhood U of the identity there exists an n such that* $\mathrm{supp}(\delta_n) \subset U$.
3) $\delta_n(x^{-1}) = \delta_n(x)$.
4) $\delta_n(x) \geq 0$ *and* $\int_G \delta_n(x)dx = 1$.

We call (δ_n) a *Dirac sequence.*

5.5 Lemma. *Let (δ_n) be a Dirac sequence. Then $\pi(\delta_n)$ converges to the identity in the sense*

$$\lim_{n\to\infty} \|\pi(\delta_n)h - h\| = 0.$$

(This means pointwise convergence.)

Proof. We have

$$\|\pi(\delta_n)h - h\| \leq \int_G \delta_n(x)\|\pi(x)h - h\|.$$

Let $\varepsilon > 0$. For n big enough we have $\|\pi(x)h - h\| < \varepsilon$ for all $x \in U_n$. We obtain $\|\pi(\delta_n)h - h\| < \varepsilon$. □

There is an obvious generalization. Let $g \in G$ then from Lemma 5.5 we see that $\pi(f_n) \circ \pi(g) \to \pi(g)$ (pointwise) A simple calculation shows

$$\pi(f) \circ \pi(g) = \pi(\tilde{f}) \quad \text{where} \quad \tilde{f}(x) = \Delta(g)f(xg^{-1}).$$

This shows the following result.

5.6 Lemma. *Let $G \to \mathrm{GL}(H)$ be a unitary representation and let $W \subset H$ be a closed subspace. Assume that there exists a subalgebra $A \subset \mathcal{C}_c(G)$ that contains a Dirac sequence and that is invariant under translation $f(x) \mapsto f(xg)$ for all $g \in G$ and such that W is invariant under A. Then W is invariant under G.*

As an application of Lemma 5.6 we get the following lemma.

5.7 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. A closed subspace $W \subset H$ is invariant under G if and only if it is invariant under $\mathcal{C}_c(G)$.*

Schur's lemma now can be formulated also for group representations.

5.8 Theorem (Schur's lemma for group representations). *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representation of a locally compact group. Every linear and continuous operator $T : H \to H$ which commutes with all $\pi(g)$, $g \in G$, is a multiple of the identity.*

Corollary. *If G is abelian then then H is one-dimensional.*

There is another consequence of Schur's lemma.

5.9 Lemma. ***Uniqueness of the scalar product.***

Algebra Version: *Let A be a star algebra and $\pi : A \to \mathrm{End}(H)$ an irreducible $*$-algebra representation on a Hilbert space H. Each inner product of H for which π is a $*$-algebra representation is a constant multiple of the original one.*

Group Version: *Let G be a locally compact group and $\pi : A \to \mathrm{End}(H)$ an irreducible unitary representation on a Hilbert space. Each inner product of H for which π is a unitary representation is a constant multiple of the original one.*

Proof. We use the Riesz lemma that states that every continuous linear form $L : H \to \mathbb{C}$ is of the form $L(x) = \langle x, a\rangle$ with a uniquely determined $a \in H$. Assume now that two scalar products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ with the property formulated in Lemma 5.9 are given. The Riesz lemma shows that there is a linear operator $T : H \to H$ with the property

$$\langle a, b\rangle_2 = \langle Ta, b\rangle_1.$$

It is easy to check that T commutes with all $\pi(a)$. Hence T is the multiplication with a number which gives the desired result. □

6. Complete reducibility

In this section we consider only unitary representations. Recall that two unitary representations $\pi : G \to \mathrm{GL}(H)$, $\pi : G \to \mathrm{GL}(H')$, are called isomorphic (as unitary representations) if there exists an isomorphism $\sigma : H \to H'$ of Hilbert spaces such that $\sigma(\pi(g)h) = \pi'(g)(\sigma(h))$.

Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. We say that another unitary representation of G occurs in π if it is isomorphic (in the unitary sense) to a sub-representation of π.

6.1 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representations and A, B be two invariant closed subspaces. Assume the the restriction of π to A is (topologically) irreducible. Then either A is orthogonal to B or the representation $\pi|A$ occurs in $\pi|B$.*

Corollary. *If both A and B are irreducible then either they are orthogonal or isomorphic (as G-representations).*

Proof. We consider the pairing $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{C}$. We first notice that it is non degenerate in the following sense. For each $a \in A$ their exists a $b \in B$ such that $\langle a, b \rangle \neq 0$ and conversely. This is clear since the orthogonal complement of B intersected with A is a closed invariant subspace. Next we construct a linear map $f : A \to B$. By the Lemma of Riesz there exists for each $a \in A$ a unique $f(a)$ in B such that $\langle a, b \rangle = \langle f(a), b \rangle$ for all $b \in B$. One easily checks that this is an intertwining operator. $\square$

6.2 Definition. *A unitary representation $\pi : G \to \mathrm{GL}(H)$ is called* ***completely reducible*** *if H can be written as the direct Hilbert sum of pairwise orthogonal closed invariant subspaces*

$$H = \widehat{\bigoplus_i} H_i$$

which are irreducible as G-representations.

In general we denote by $\hat{G}$ the set of all isomorphy classes of irreducible unitary representations of G and call it the *unitary dual* of G. Recall that each irreducible unitary representation $\pi : G \to \mathrm{GL}(H)$ is one dimensional if G is abelian. Hence it is of the form $\pi(g)(h) = \chi(g)h$ where χ is a character of G. By definition, this is a continuous homomorphism from G into the group of complex numbers of absolute value 1. Unitary isomorphic representations give the same character. This gives a bijection with $\hat{G}$ and the set of all unitary characters. Characters can be multiplied in an obvious way. Hence, for abelian G, the set $\hat{G}$ is a group as well. One can show that it carries a structure as locally compact group.

6.3 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ a unitary representation which is completely reducible,*

$$H = \widehat{\bigoplus_{i \in I}} H_i, \qquad H_i \subset H.$$

Let $\tau \in \hat{G}$. Then

$$H(\tau) = \widehat{\bigoplus_{i \in I,\ \pi_i \in \tau}} H_i$$

is the closure of the sum of all irreducible closed invariant subspaces of H that are of type τ. In particular, it is independent of the choice of the decomposition.

This follows immediately from Lemma 6.1. □

We call $H(\tau)$ the τ-isoptypic component of π. This is well-defined. The irreducible components H_i are usually not well-defined. Look at the example of the group G that consists only of the unit element.

7. Generalities about compact groups

In this section we treat some general facts about representations of compact groups. Readers who are mainly interested in the classification of the irreducible unitary representations of the group $\mathrm{SL}(2,\mathbb{R})$ can skip this section, since the only compact group which occurs in this context is the group $\mathrm{SO}(2,\mathbb{R})$. This group is not only compact but also abelian which makes the theory rather trivial.

We need some results of functional analysis. We recall the notion of equicontinuity:

7.1 Definition. *A set $\mathcal{M}$ of functions on a topological space X is called equicontinuous at a point $a \in X$ if for any point $\varepsilon > 0$ their exists a neighborhood U of a such that*

$$|f(x) - f(a)| < \varepsilon \quad \textit{for all} \quad x \in U,\ f \in \mathcal{M}.$$

The set is called equicontinuous if this is the case at all $a \in X$.

(The point is the independence of the neighborhood U from f.) We recall a basic result from functional analysis.

7.2 Theorem (theorem of Arzela-Ascoli). *Let X be a locally compact space with countable basis of the topology. Let $\mathcal{M}$ be an equicontinuous set of functions on X such that the set of numbers $f(x)$, $f \in \mathcal{M}$, is bounded for every $x \in X$. Then each sequence of $\mathcal{M}$ admits a subsequence that converges locally uniformly on X.*

There are variants of this theorem in which equicontinuity does not appear. Let for example $X \subset \mathbb{R}^n$ be an open subset and assume that $\mathcal{M}$ is a set of *differentiable* functions such that there exists a constant C such that

$$|f(x)| \le C \quad \text{and} \quad |(\partial f/\partial x)(x)| \le C \quad \text{for all } x \in X.$$

Then the mean value theorem of calculus shows that this set is equicontinuous.

Another main tool will be the spectral theorem for compact operators on Hilbert spaces. Let H be a Hilbert space. A linear and continuous operator $T : H \to H$ is called *compact* if the image any bounded set is contained in a compact set. For example this is the case if the image of T is finite dimensional. The identity is compact if and only if H is finite dimensional. The set of all compact operators is closed under the operator norm. So, let $T_1, T_2, \dots$ be a sequence of compact operator and T another bounded operator such that $\|T_n - T\|$ tends to 0. Then T is compact.

Recall that an operator T is called normal if it commutes with its adjoint, $T \circ T^* = T^* \circ T$.

7.3 Theorem (Spectral theorem for compact operators). *Let $T : H \to H$ be a compact and normal operator. The set of eigenvalues is either finite or it is countable and 0 is the only accumulation point of it. The eigenspaces $H(T, \lambda)$ are pairwise orthogonal and for $\lambda \neq 0$ they are finite dimensional. The sum of all eigenspaces is dense in H. Hence we have a Hilbert space decomposition*

$$H = \widehat{\bigoplus_\lambda} H(T, \lambda).$$

We will not proof this theorem here.

We give an example of a compact operator.

7.4 Proposition. *Let X be a compact topological space and dx a Radon measure. Let $K \in \mathcal{C}(X, X)$ be a continuous function. The operator*

$$L_K : L^2(X, dx) \longrightarrow L^2(X, dx), \quad L_K(f)(x) := \int_X K(x, y) f(y) dy.$$

is a compact (continuous and linear) operator.

We mention that every square integrable function f on a compact space is integrable (since one can write $f = 1 \cdot f$ as product of two square integrable functions). Since $K(x, y)$ for fixed x is an L^2-function the existence of the integral in Proposition 7.4 is clear. Clearly the functions $L_K f$ are continuous. Even more we have

$$|L_X(f)(x)| \leq c \|f\|_2$$

with some constant c by the Cauchy-Schwarz inequality. This also implies that $L_X f \in L^2(X, dx)$ and moreover

$$\|L_K f\|_2 \leq C \|f\|_2$$

with some constant C. Hence the operator is linear and also continuous.

But we have a stronger property. It is easy to show that the set of functions

$$\{L_K f; \quad f \in L^2(X, dx), \quad \|f\|_2 \leq 1\}$$

is equicontinuous. This implies that L_K is a compact operator. For this we have to prove the following. Let $f_n \in L^2(X, dx)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$. We have to show that $L_K f_n$ has a sub-sequence that converges in $L^2(X, dx)$. The theorem of Arzela-Ascoli shows that $L_K f_n$ converges uniformly. Hence it converges point-wise and and all functions are bounded by a joint constant. Since X is compact, constant functions are integrable and we can apply the Lebesgue limit theorem to obtain convergence in $L^2(X, dx)$. □

7.5 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation of a locally compact group G on a Hilbert space G. Assume that there exists a Dirac sequence $\delta_n \in \mathcal{C}_c(G)$ such that all $\pi(\delta_n)$ are compact operators. Then the representation decomposes into irreducibles with finite multiplicities.*

Proof. We consider pairs that consist of a closed invariant subspace $H' \subset H$ such the restriction of π to H' is completely reducible and a distinguished decomposition $H' = \widehat{\bigoplus}_{i \in I} H'_i$ into irreducibles. We define an ordering for such pairs. The pair $H' = \widehat{\bigoplus}_{i \in I} H'_i$ is less or equal than the pair $H'' = \widehat{\bigoplus}_{j \in J} H''_j$ if each space H'_i equals some H''_j. (Especially $H' \subset H''$). From Zorn's lemma easily follows that there exists a maximal member. We call its orthogonal complement U. This space cannot contain any irreducible subspace since this could be used to enlarge the maximal element. Hence we have to show:

let π be a representation as in the proposition which is not zero. Then there exists at least one irreducible closed subspace.

To prove this we choose an element f of the Dirac sequence such that $\pi(f)$ is not identically zero. This element will kept fixed during the proof. We also choose an eigenvalue $\lambda \neq 0$ of $\pi(f)$ Let $H(f, \lambda) \subset H$ the eigenspace. This is a finite dimensional vector space.

There may be other invariant closed subspaces which have a non-zero intersection with $H(f, \lambda)$. We choose a a closed subspace E such that the dimension of its intersection with $H(f, \lambda)$ is non-zero and minimal. Then we set $W = E \cap H(f, \lambda)$. There still may exist several closed invariant subspaces H that share with E the property $W = F \cap H(f, \lambda)$. We take the intersection of all these F and get in this way a smallest closed invariant subspace $F \subset E$ with $W = F \cap H(f, \lambda)$. We claim that this F is irreducible. For this we take any orthogonal decomposition $F = A \oplus B$. The eigenvalue λ must occur as eigenvalue of λ in one of the spaces A, B. (The restriction of a compact operator to a closed invariant subspace remains compact and hence decomposes into eigen spaces.) Let us assume that it occurs in A. Then $A \cap H(f, \lambda)$ is not zero. It must agree with W because of the minimality property of $\dim W$. Moreover it must agree with F because of the minimality property of F. This shows the irreducibility.

It remains to prove that the multiplicities are finite. Let $\tau \in \hat{G}$. Let $H_1, \dots, H_m$ be pairwise orthogonal invariant closed subspaces of type τ. We claim that m is bounded. There exists an element $f = \delta_n$ from the Dirac sequence such that $\pi(f)$ is not zero on H_1. There exists a non-zero eigenvalue λ. This eigenvalue then occurs in all H_i since they are all isomorphic (as representations). Since the multiplicity of the eigenvalue is finite the number m must be bounded. □

A special case of Proposition 7.5 gives the following basic result.

7.6 Theorem. *Let K be a compact group. The regular representation of K on $L^2(K)$ (translation from the right) is completely reducible with finite multiplicities.*

Another basic result for compact groups is the following theorem.

7.7 Theorem. *Every irreducible unitary representation of a compact group is finite dimensional*

We also mention the following result for compact groups.

7.8 Proposition. *Every continuous finite dimensional representation $\pi : K \to \mathrm{GL}(H)$ of a compact group K is unitarizable. This means that there exists a hermitian scalar product on H such that π is unitary.*

Chapter II. The special linear group of degree two

1. The simplest compact group

We study the group

$$K = \mathrm{SO}(2, \mathbb{R}).$$

So K consists of all real 2×2 matrices k of determinant 1 with the property

$$k'k = e.$$

Because of

$$k^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \left(k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

this means that k is of the form

$$k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^2 + b^2 = 1.$$

For $k \in K$ the complex number $\zeta = a + ib$ is of absolute value 1. Recall that the set of all complex numbers of absolute value 1 is a group under multiplication. One easily checks that the map

$$\mathrm{SO}(2, \mathbb{R}) \xrightarrow{\sim} S^1, \quad k \longmapsto \zeta,$$

is an isomorphisms of locally compact groups. So we see that K is a compact and abelian group. Hence we know that each irreducible unitary representation is one-dimensional and corresponds to a character. The characters of S^1 are easy. The correspond to the integers $\mathbb{Z}$. For each integer n we can define

$$\chi_n(k) = \chi_n(\zeta) := \zeta^n.$$

For an arbitrary unitary representation $\pi : K \to \mathrm{GL}(H)$ we can consider the corresponding isotypic component

$$H(n) := \{h \in H; \quad \pi(g)(h) = \chi_n(g)h\}.$$

Another way to write the elements of $\mathrm{SL}(2, \mathbb{R})$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Here ϕ is determined mod $2\pi\mathrm{i}\mathbb{Z}$. The character χ_n in this presentation is given by

$$\chi_n(k) = e^{2\pi\mathrm{i}k}.$$

2. The Haar measure on the special linear group of degree two

We use the following notations:

$$\begin{aligned} &G = \mathrm{SL}(2,\mathbb{R}),\\ &A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; \quad t \in \mathbb{R} \right\},\\ &N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; \quad x \in \mathbb{R} \right\},\\ &K = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad \theta \in \mathbb{R} \right\}. \end{aligned}$$

2.1 Lemma (Iwasawa decomposition). *The map*

$$A \times N \times K \longrightarrow G, \quad (a,n,k) \longmapsto ank,$$

is topological.

Proof. The elements of K act as rotations on $\mathbb{R}^2$. To any $g \in G$ one can find a rotation k such that gk fixes the x-axis. Then gk is triangular matrix. This gives the prove of the lemma. □

One can write the decomposition explicitly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \frac{1}{\sqrt{c^2+d^2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & ac+bd \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

We denote the Haar measures on A, N, K by da, dn, dk. Since $t \mapsto a_t$ is an isomorphism of groups, we have $da = dt$ where dt denotes the standard measure of $\mathbb{R}$. The measure dk is normalized such that the volume of K is 1.

We first consider the group $P = AN$ of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ with positive diagonal. The map $A \times N \to P$ is topological (but not a group isomorphism). Recall that

$$\int_P f(p)dp := \int_A \int_N f(an)dnda$$

is a Haar measure (Lemma I.3.2).

2.2 Proposition. *A Haar measure on $G = \mathrm{SL}(2,\mathbb{R})$ can be obtained as follows*

$$\int_G f(x)dx = \int_A \int_N \int_K f(ank)\, dk\, dn\, da.$$

Proof. Since K is compact we have $\Delta_G|K = \Delta_K$. Hence the invariant quotient measure On $K\backslash G$ exists. There is a natural topological map $P \to K\backslash G$. The quotient measure gives a Haar measure on P. The rest comes from defining properties of a quotient measure (Proposition 1.3.3). □

3. The space $S_{m,n}$

We consider the groups

$$G = \mathrm{SL}(2,\mathbb{R}) \quad \text{and} \quad K = \mathrm{SO}(2,\mathbb{R}).$$

Making use of the Iwasawa decomposition, we can write any function $f : G \to \mathbb{R}$ as functions of the variables a, n, θ

$$f(g) = g(a, n, \theta).$$

Since g can be considered as a function on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$, it makes sense to talk about differentiable g and in this way of differentiable f. We denote the subspace of differentiable functions of $\mathcal{C}_c(G)$ by $\mathcal{C}_c^\infty(G)$.

3.1 Definition. *The space $S_{m,n}$ consists of all $f \in \mathcal{C}_c^\infty(G)$ with the property*

$$f(k_\theta x k_{\theta'}) = f(x)e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'} \qquad (x \in G).$$

Let be $f \in \mathcal{C}_c^\infty(G)$. Then the Fourier coefficient

$$f_{m,n}(x) = \int_0^{2\pi}\int_0^{2\pi} f(k_\theta x k_{\theta'})e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'}\,d\theta d\theta'$$

is contained in $S_{m,n}$. From the theory of Fourier series we obtain

$$f(x) = \sum_{m,n} f_{m,n}(x)e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'}$$

where the convergence is absolute and locally uniform in x. Let $\mathrm{supp}(f)$ be the support of f. Then $K\,\mathrm{supp}(f)K$ contains the support of $f_{m,n}$. We have proved the following result:

3.2 Lemma. *Let be $f \in \mathcal{C}_c^\infty(G)$ and let be $\varepsilon > 0$. There exists a function g which is a finite linear combination from functions contained in $S_{m,n}$ and with the following property:*

a) $\operatorname{supp}(g) \subset K \operatorname{supp}(f) K$,
b) $|f(x) - g(x)| < \varepsilon$ *for* $x \in G$.

Corollary. *The algebraic sum $\sum_{m,n} S_{m,n}$ is dense in the space $L^1(G, dx)$ with respect to the norm $\|\cdot\|_1$.*

Here dx of course is a Haar measure. Recall that G is a unimodular group, hence we have to define

$$f^*(x) = \overline{f(x^{-1})}.$$

We study the convolution.

3.3 Lemma. *We have*

a) $S_{m,n} * S_{p,q} = 0$ *if* $n \neq p$.
b) $S_{m,n}^* = S_{n,m}$.
c) $S_{m,n} * S_{n,q} \subset S_{m,q}$.

The proof can be given by an easy calculation which can be left to the reader. □

From Lemma 3.3 we see that $S_{n,n}$ is a star algebra.

3.4 Proposition. *The algebra $S_{n,n}$ is commutative.*

Now we consider a representation of $G = \mathrm{SL}(2, \mathbb{R})$,

$$\pi : G \longrightarrow \mathrm{GL}(H).$$

At the moment, it is enough to assume that H is a Banach space. We restrict this representation to K and consider the (closed) subspace

$$H(n) := \{h \in H; \quad \pi(k)(h) = \chi_n(k)h\}.$$

It is clear that the algebraic sum of all $H(n)$ is a direct sum. In the case that π is a unitary representation we know more, namely that the $H(n)$ are pairwise orthogonal and that H is the direct Hilbert sum of the $H(n)$. For an element h in the algebraic sum, we denote by h_n the component in $H(n)$.

3.5 Lemma. *The algebra $S_{m,n}$ maps H into $H(m)$. It maps $H(q)$ to zero of $n \neq q$.*

Corollary. *The algebraic sum $\mathcal{A} = \sum S_{m,n}$ acts on the algebraic sum $\sum H_m$.*

Let $h \in H(n)$ be a non-zero element. We consider the space $\mathcal{A}h$. The algebra $\mathcal{A}$ is L^1-dense in $\mathcal{C}_c(G)$. Hence $\mathcal{A}h$ is a dense subspace of H. It is contained in the algebraic sum $\sum H(m)$. We can consider the projection of $\sum H(m)$ to $H(n)$. The image of $\mathcal{A}h$ is dense in $H(n)$. Lemma 3.5 shows that this image equals the image of $S_{n,n}h$. This shows that $S_{n,n}$ acts topologically irreducible on $H(n)$.

3.6 Proposition. *The algebra $S_{n,n}$ acts topologically irreducibly on $H(n)$ if this space is not zero.*

Since $S_{n,n}$ is abelian, we now obtain the following theorem.

3.7 Theorem. *Let π by an irreducible representation on a Banach space. Then the algebraic sum of the $H(n)$ is dense in H. Assume that $H(n)$ is finite dimensional. Then* $\dim H(n) \leq 1$. *This is always the case if π is unitary (and in this case H is the direct Hilbert sum of the $H(n)$).*

We just mention that this a special case of a more general result that holds for any semi simple Lie group G and a maximal compact subgroup. Examples are $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n, \mathbb{R})$. For every irreducible unitary representation of G the K-isotypic components are finite dimensional. In other words: each irreducible unitary representation of K (which is always finite dimensional) occurs with finite multiplicity in $\pi|K$. The proof more involved, mainly since K is not commutative in general.

A vector $h \in H$ is called K-finite If the space generated by all $\pi(k)h$ is finite dimensional. The space of K-finite vectors is denoted by $H(K)$. The elements of $H_{m,n}$ are K-finite. Since every finite dimensional representation of a compact group is completely reducible, we obtain the following description.

3.8 Lemma. *Let π be a representation of G on a Banach space H. Then*

$$H(K) = \sum_{m \in \mathbb{Z}} H(m) \qquad \textit{(algebraic sum).}$$

It is important to describe for a given irreducible unitary representation π the set of all n sich such that $H(n)$ is different from zero (and then one-dimensional). For this we look for operators that shift $H(n)$ which means that $H(n)$ is mapped into another $H(m)$. We will find such operators in the Lie algebra.

4. The Lie-algebra

We recall the exponential function for matrices $A \in \mathbb{C}^{(n,n)}$:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

It is clear that this series converges absolutely. The rule

$$\exp(A + B) = \exp(A) + \exp(B)$$

holds if A, B commute. The rule

$$B^{-1}\exp(A)B = \exp(B^{-1}AB$$

is trivial. We need also the rule

$$\det\exp(A) = \exp(\operatorname{tr}(A))$$

which can be reduced to diagonal matrices (using the previous rule and the fact that the set of all matrices with n pairwise different eigenvalues is dense in the set of all matrices). We use the notation (Lie-bracket)

$$[A, B] = AB - BA.$$

Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup. We consider the set $\mathfrak{g}$ of all matrices A such that $\exp(tA) \in G$ for all $t \in \mathbb{R}$. It can be shown that $\mathfrak{g}$ is a Lie-algebra. This means that $\mathfrak{g}$ is a vector space and that $A, B \in \mathfrak{g}$ implies that $[A, B] \in \mathfrak{g}$. One can show that G is a smooth subset and that

$$\exp : \mathfrak{g} \longrightarrow G$$

is a local diffeomorphism at the origin. We do not need this general theory, since these facts can be verified directly for $G = \mathrm{SL}(2, \mathbb{R})$. From now on we restrict to this case.

In this case we have

$$\mathfrak{g} = \{A \in \mathbb{R}^{(2,2)}; \quad \operatorname{tr}(A) = 0\}.$$

Notice that it is trivial that this is a Lie-algebra. Besides $\mathfrak{g}$ we need also its complexification

$$\mathfrak{g}_{\mathbb{C}} := \{A \in \mathbb{C}^{(2,2)}; \quad \operatorname{tr}(A) = 0\}.$$

This is a complex Lie algebra (i.e. it is a complex vector space and invariant under the Lie-bracket).

5. The derived representation

Differential calculus usually is defined for maps $U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open subset. There is a straight forward generalization where $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ are replaced by Banach spaces, where in this context they are understood as Banach spaces over the field of real numbers. It is clear what this means. A map $f : U \to F$ in this context is called differentiable at $a \in U$ if there exists a continuous (real) linear map $L_a : E \to F$ such that

$$f(x) - f(a) = L_a(x - a) + r(x) \quad \text{where} \quad \lim_{x \to a} \frac{\|r(x)\|}{\|x - a\|} = 0.$$

If this is true for every $a \in U$ we call f differentiable. Then we can consider the derivative

$$Df : U \longrightarrow \mathrm{Hom}(E, F), \quad df(a) = L_a.$$

Since $\mathrm{Hom}(E, F)$ is a Banach space too, we can ask for differentiability of df. In this way one can define the space of infinite differentiable functions $\mathcal{C}^\infty(U, F)$. As in the finite dimensional case, the chain rule holds for (infinitely often) differentiable functions. We also mention that a continuous linear map is differentiable by trivial reasons.

We want apply this to functions $G \to H$ where H is a Banach space (as usual over the complex numbers). Assume that $\pi : G \longrightarrow \mathrm{GL}(H)$ be a continuous representation. We associate to an arbitrary vector $h \in H$ a function

$$G \longrightarrow H, \quad x \longmapsto \pi(x)h.$$

We call the vector h differentiable if this function is infinitely often differentiable. We denote the space of differentiable vectors by H^∞. These is a sub-vector space. It depends of course on π. Hence, for example, H^∞_π is a more careful notation.

We give examples of an differentiable vector.

5.1 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and $f \in \mathcal{C}_c^\infty(G)$. Then the image of $\pi(f)$ is contained in H^∞.*

Corollary. *Let m be an integer such that $H(m)$ is finite dimensional. Then the elements of $H(m)$ are differentiable.*

Let $X \in \mathfrak{g}$ and $h \in H^\infty$. The map

$$\mathbb{R} \longrightarrow H, \quad t \longmapsto \pi(\exp(tX)h$$

is differentiable, since it is the composition of two differentiable maps. Hence we can define the operator $d\pi(X) : H^\infty \to H$:

$$d\pi(X)h := \frac{d}{dt}\pi(\exp(tX)h)\Big|_{t=0}.$$

This is related to a another construction, the Lie-derivative (from the left). This is for each $X \in \mathfrak{g}$ a map

$$\mathcal{L}_X : \mathcal{C}^\infty(G, H) \longrightarrow \mathcal{C}^\infty(G, H)$$

which is defined by

$$\mathcal{L}_X f(a) = \frac{d}{dt} f(a \exp(tX))\Big|_{t=0}.$$

(It is easy to show that $\mathcal{L}_X f$ is differentiable.) The Lie-derivative has nothing to do with the representation π. But we get a link to the derived representation if we apply it to functions of the type $x \mapsto \pi(x)h$.

5.2 Lemma. *Let $X \in \mathfrak{g}$ and $h \in H^\infty$. We consider the differentiable function $f(x) = \pi(x)h$ on G. The the formula*

$$\pi(a)d\pi(X)h = (\mathcal{L}_X f)(a)$$

holds, in particular

$$d\pi(X)h = (\mathcal{L}_X f)(e).$$

Proof. The second formula is just true by definition. The first one can be obtained if one applies $\pi(a)$ to the second one. One just has to observe that $\pi(a)$ commutes by continuity with the limit

$$\lim_{t\to 0} \frac{\pi(\exp)tX)h - h}{t}. \qquad \square$$

6. Explicit formulae for the Lie-derivatives

In the following we use the basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathfrak{g}$. We also will consider the complexification $\mathfrak{g}_\mathbb{C}$. Here we use the (complex) basis

$$W, \quad E^- = H - \mathrm{i}V, \quad E^+ = H + \mathrm{i}V.$$

So we have

$$E^- = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}.$$

We extend the Lie-derivatives to $\mathfrak{g}_\mathbb{C}$ by $\mathbb{C}$-linearity

$$\mathcal{L}_{A+\mathrm{i}B} = \mathcal{L}_A + \mathrm{i}\mathcal{L}_B.$$

This is possible, since H and hence $\mathcal{C}^\infty(G, H)$ is a complex vector space

From the Iwasawa decomposition we know that we can write $g \in G$ in the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \sqrt{y}\, x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with unique x and $y > 0$. The angle θ is determined mod 2π. We need the expressions for x, y, θ in terms of a, b, c, d. To get them it is useful to use complex numbers. Let τ be a complex number in the upper half plane, $\operatorname{Im}\tau > 0$. Since

c, d are real but not both zero, the number $c + d\tau$ is different from zero. Hence we can define

$$g(\tau) = \frac{a\tau + b}{c\tau + b}.$$

Let h be a second matrix from G. A direct computation which we omit shows

$$(gh)(\tau) = g(h(\tau)).$$

We also notice

$$g_\theta(\mathrm{i}) = \mathrm{i}.$$

Hence we obtain

$$\frac{a\mathrm{i} + b}{c\mathrm{i} + d} = x + \mathrm{i}y.$$

This gives us x and y in terms of a, b, c, d. For example one gets

$$y = \frac{1}{c^2 + d^2}.$$

Looking at the second row of the Iwasawa decomposition we get

$$c\sqrt{y} = -\sin\theta, \quad d\sqrt{y} = \cos\theta.$$

This shows

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta = \frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

This gives as

$$\theta = \operatorname{Arg} \frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

Since θ is only determined mod 2π, we have to say a word about the choice of the argument Arg. In the following we will fix $g \in G$ and $X \in \mathfrak{g}$ and consider

$$g(t) = g\exp(tX)$$

for small t. We write $x(t), y(t), \theta(t)$ in this case. It is clear that for small t the function $\theta(t)$ can be chosen such that it depends differentially on t. If we insert $t = 0$ we get the original x, y, θ.

For the Lie-derivative we have to consider a differentiable function f on G. We can write it as function f of three variables. We get

$$f(g(t)) = F(x(t), y(t), \theta(t)).$$

By means of the chain rule we get

$$\frac{d}{dt} f(g(t)) = \frac{\partial F}{\partial x}\dot{x}(t) + \frac{\partial F}{\partial y}\dot{y}(t) + \frac{\partial F}{\partial \theta}\dot{\theta}(t).$$

Recall that we have to evaluate this expression at $t = 0$ to get the Lie derivative.

As an example we take

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$g(t) = \begin{pmatrix} a & b + ta \\ c & d + tc \end{pmatrix}.$$

We obtain

$$z(t) = \frac{a\mathrm{i} + at + b}{c\mathrm{i} + ct + d}.$$

Differentiation and evaluating at $t = 0$ gives

$$\dot{z}(0) = \frac{1}{(c\mathrm{i} + d)^2}.$$

Using the formulae

$$y = \frac{1}{c^2 + d^2}, \quad e^{2\mathrm{i}\theta} = \frac{(d - \mathrm{i}c)^2}{c^2 + d^2}$$

we obtain

$$\dot{z}(0) = ye^{2\mathrm{i}\theta} \quad \text{or} \quad \dot{x}(0) = y\cos 2\theta, \ \dot{y}(0) = y\sin 2\theta.$$

Finally, to compute $\dot{\theta}(0)$, we use the formula

$$\cos\theta(t) = (d + ct)\sqrt{y(t)}.$$

Differentiation gives

$$-\dot{\theta}(t)\sin\theta(t) = (d + ct)\frac{\dot{y}(t)}{2\sqrt{y(t)}} + c\sqrt{y(t)}.$$

Evaluating by $t = 0$ we get

$$\dot{\theta}(0)\sin\theta = \frac{d\dot{y}(0)}{2\sqrt{y}} - c\sqrt{y}.$$

We insert $-c\sqrt{y} = \sin\theta$ and $\dot{y}(0) = y\sin 2\theta = 2y\sin\theta\cos\theta$ to obtain

$$\dot{\theta}(0) = -d\sqrt{y}\cos\theta + 1 = -\cos^2\theta + 1 = \sin^2\theta.$$

Another – even easier example – is $\mathcal{L}_W$. A simple computation gives

$$\exp tW = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence we obtain that $\mathcal{L}_W$ is given by the operator $\partial/\partial\theta$. In a similar way other elements of the Lie algebra can be treated. Since $X = 2X - W$ we get now $\mathcal{L}_V$. We omit the computation for H and just collect the formulae together.

6.1 Proposition. *Let $f \in \mathcal{C}^\infty$ and $A \in \mathfrak{g}$. We denote by $F(x,y,\theta)$ the corresponding function in the coordinates and similarly $G(x,y,\theta)$ for $g = \mathcal{L}_A f$. The operator $F \mapsto G$ can be described explicitly as follows:*

$$\begin{aligned}
\mathcal{L}_X &= y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + \sin^2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_W &= \frac{\partial}{\partial\theta},\\
\mathcal{L}_V &= y2\cos 2\theta\frac{\partial}{\partial x} + 2y\sin 2\theta\frac{\partial}{\partial y} - \cos 2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_H &= -2y\sin 2\theta\frac{\partial}{\partial x} + 2y\cos 2\theta\frac{\partial}{\partial y} + \sin 2\theta\frac{\partial}{\partial\theta},
\end{aligned}$$

and, as a consequence,

$$\mathcal{L}_{E^-} = -2\mathrm{i}ye^{-2\mathrm{i}\theta}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big) + \mathrm{i}e^{-2\mathrm{i}\theta}\frac{\partial}{\partial\theta}.$$

7. Analytic vectors

Let E, F be Banach spaces over the field of real numbers and let $U \subset E$ be an open subset. We introduced the notion of a differentiable map $U \to F$ In the case that E is finite dimensional (but F may be not) we can also define the notion of an analytic map. In the case $E = \mathbb{R}^n$ this means as usual that for each $a \in U$ there exists a small neighbourhood in which there exists an absolutely convergent expansion as power series

$$f(x) = \sum_{\nu\in\mathbb{N}_0^n} a_\nu (x_1 - a_1)^{\nu_1}\cdots(x_n - a_n)^{\nu_n} \qquad (a_\nu \in F).$$

This notion is invariant under linear transformation of the coordinates, hence it carries over to arbitrary E. We denote by $\mathcal{C}^\omega(U,F)$ the space of all analytic functions. This is a subspace of $\mathcal{C}^\infty(U,F)$. The basic property of analytic functions is the principle of analytic continuation. Assume that U is connected and that $a \in U$ a point that all derivatives of f or arbitrary order vanish (this is understood to include $f(a) = 0$). Then f is identically zero.

Using the standard coordinates of G, we can define the notion of analytic function $G \to H$ into any Banach space. If $\pi : G \to \mathrm{GL}(H)$ is a representation we can define the notion of an analytic vector $h \in H$. By definition this means that the function $\pi(x)h$ on G is analytic. The set H^ω of all analytic vectors is a sub-vector space of H^∞.

We recall the formula for the Lie-derivative

$$(\mathcal{L}_X f)(y) = \frac{d}{dt} f(y \exp(tX))\Big|_{t=0}.$$

We replace y by $y \exp(uX)$ and obtain

$$(\mathcal{L}_X f)(y \exp(uX)) = \frac{d}{dt} f(y \exp((u+t)X))\Big|_{t=0} = \frac{d}{du} f(y \exp(uX)).$$

By induction follows

$$(\mathcal{L}_X^n f)(y \exp(uX)) = \frac{d^n}{du^n} f(y \exp(uX)).$$

The Taylor expansion of the function $t \mapsto f(y \exp(tX))$ is given by

$$\begin{aligned} f(y \exp(tX)) &= \sum_{n=0}^{\infty} \frac{1}{n!} \, \frac{d^n}{dt^n} f(y \exp(tX))\Big|_{t=0} t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \, (\mathcal{L}_X^n f)(y) \, t^n. \end{aligned}$$

This formula is true for given X, y if $|t|$ is sufficiently small, $|t| < \varepsilon$. For a real constant the formula $\mathcal{L}_{cX} = c\mathcal{L}_X$ can be checked. This shows that (for fixed y) the Taylor formula holds if X is in a sufficiently small neighborhood of the origin. We specialize the Taylor expansion to the function $f(x) = \pi(x)h$ and to $t = 1$.

7.1 Proposition. *For sufficiently small X the formula*

$$\pi(\exp(X))h = \sum_{n=0}^{\infty} d\pi(X)^n h$$

holds.

We now obtain the following important result.

7.2 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and let $V \subset H$ be a linear subspace consisting of analytic vectors that is invariant under $d\pi(\mathfrak{g})$. Then the closure of V is invariant under G.*

8. The Casimir operator

Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and

$$\varrho : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathcal{A}$$

be a linear map with the property

$$\varrho([A, B]) = \varrho(A)\varrho(B) - \varrho(B)\varrho(A).$$

Our typical example is that $\mathcal{A}$ is the algebra of (algebraic) endomorphisms of an abstract vector space $\mathcal{H}$. In this case we talk about a Lie-algebra representation on $\mathcal{H}$. We denote the image of element $A \in \mathfrak{g}_{\mathbb{C}}$ by the corresponding bold letter $\boldsymbol{A}$. We define the Casimir element by

$$\omega = \boldsymbol{H}^2 + \boldsymbol{V}^2 - \boldsymbol{W}^2.$$

The basic property of the Casimir element is that it commutes with the image of $\mathfrak{g}_{\mathbb{C}}$.

8.1 Lemma. *The Casimir element ω (with respect to $\varrho : \mathfrak{g} \to \mathcal{A}$) commutes with all $\boldsymbol{A}$ for $A \in \mathfrak{g}$.*

The Lie algebra $\mathfrak{g}$ acts in the space $\mathcal{C}^\infty(G)$. Hence we can consider the Casimir operator ω acting on this space. Using the formulae on Proposition 5.2 we get the explicit expression

$$\omega = 4y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 4y\frac{\partial^2}{\partial x \partial y}.$$

Similar to the Laplace operator, this is an elliptic differential operator. We make use of a basis result that the solutions of elliptic differential equations with analytic coefficients are analytic functions. We obtain that each eigen function

$$\omega f = \lambda f$$

is an analytic function. We can extend this result to differentiable functions $f : G \to H$ where H is a Banach space. Let L be continuous linear functional on H. Then one has the trivial formula

$$L\omega f = \omega L f.$$

Let f be an eigen function of ω. Then the analycity result above shows that $L \circ f$ is analytic for all L. A general fact about analytic functions states:

8.2 Proposition. *A function $f : G \to H$ is analytic if an only if $L \circ f$ is analytic for every continuous linear function L.*

Let now $\pi : G \to \mathrm{End}(H)$ a Banach representation and let $h \in H$ by a differentiable vector. Then there is the Casimir operator acting on H^∞. Assume that h is an eigen vector. We claim that h is an analytic vector. We have to show that the function $f_h(x) = \pi(x)h$ is analytic. For any $A \in \mathfrak{g}$ we have

$$\mathcal{L}_A f_h = f_{d\pi(A)h}.$$

This carries over to the Casimir operator. So we can write

$$\omega f_h = f_{\omega h}.$$

By assumption h is an eigen vector of the Casimir operator. This implies that f_h is an eigen function. Hence f_h is analytic. This means that h is analytic. This gives the following result.

8.3 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and let $h \in H$ be a differentiable vector which is ein eigen vector of the Casimir operator. Then h is analytic.*

9. Admissible representations

Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation. It is easy to check that $d\pi(W)$ acts on $H(n)$ by multiplication with $\mathrm{i}n$. Recall that $\dim H(n) \leq 1$ if π is an irreducible unitary representation. Recall also that $\mathfrak{g}$ acts irrreducibly on the algebraic sums $\mathcal{H} = \sum H(n)$. We are led to consider the following algebraic objects.

9.1 Definition. *An (algebraic) representation of $\mathfrak{g}$ on a on vector space $\mathcal{H}$ is called admissible if the following conditions are satisfied.*

1) *It is algebraically irreducible (in the sense that there is no invariant subspace different form 0 and $\mathcal{H}$).*
2) *The eigen spaces $H(n)$ of the operator W with eigen value $\mathrm{i}n$, $n \in \mathbb{Z}$ are of dimension ≤ 1.*
3) *The space $\mathcal{H}$ is the (algebraic) direct sum of the subspaces $H(n)$.*

It is clear what an isomorphism of admissible representation means. We emphasize that this is understood in a pure algebraic way. As we have seen, every irreducible unitary representation $G \to \mathrm{GL}(H)$ has an underlying admissible representation of $\mathfrak{g}$ on $\mathcal{H} = H(K)$. It is clear that unitary isomorphic representations have underlying isomorphic admissible representation. The converse is also true.

9.2 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$, $\pi' : G \to \mathrm{GL}(H')$ be two irreducible representations such that the underlying admissible representations are (algebraically) isomorphic. Then π, π' are isomorphic as unitary representations.*

We study in more detail admissible representations. For this we will use the basis E^+, E^+, W for $\mathfrak{g}_{\mathbb{C}}$. We recall

$$[E^+, E^-] = -4\mathrm{i}W, \quad [W, E^+] = 2\mathrm{i}E^+, \quad [W, E^-] = -2\mathrm{i}E^-.$$

9.3 Lemma. *For an admissible representation of $\mathfrak{g}$ we have*

$$E^+(H(n)) \subset H(n+2), \quad E^-(H(n)) \subset H(n-2).$$

From Lemma 9.3 we see that the spaces

$$H^{\mathrm{even}} = \sum_{n \text{ even}} H(n), \quad H^{\mathrm{odd}} = \sum_{n \text{ odd}} H(n)$$

are invariant subspaces. Hence we have to distinguish between and even case (all $H(2n+1)$ are zero) and an odd case (all $H(2n)$ are zero).

Let S be a set of all integers which are all odd or all zero. We call S an interval if for $m, n \in S$ each number of the same parity between m and n is contained in S. We claim now that the set S of all n such that $H(n) \neq 0$ is an interval. To prove this we consider an $n \in S$ such that $H(n)$ is different from zero. Recall that $H(n)$ is one-dimensional. We choose a generator h. The space H is generated by all $A_1 \dots A_m h$ where $A_i \in \mathfrak{g}_{\mathbb{C}}$. From the relations between the generators we see that H is generated by $E_+^m h$ and $E_-^m h$. Let for example $H(n+2k) = 0$, $k > 0$. Then $E_+^{n+2k} h = 0$ and hence all $H(m)$, $m > n + 2k$, are zero. Hence S is an interval. We are only interested in infinite dimensional representations.

9.4 Proposition. *For the set S of integers m with the property $H(m) \neq 0$ of an infinite dimensional admissible representation there are the following possibilities:*

1) *S is the set of all even integers.*
2) *S is the set of all odd integers.*
3) *There exists $n \in S$ such that S consists of all $m \geq n$ with the same parity.*
4) *There exists $n \in S$ such that S consists of all $m \leq n$ with the same parity.*

In the case 3) we call n the lowest weight and the non-zero elements of $H(n)$ the lowest weight vectors. Similarly we call in the case 4) n the highest weight.

We study case 1) in more detail. We choose a non-zero vector $h \in H(0)$. We know that E^+ is non zero on all $H(n)$ (n even). Hence we can define for all even n a uniquely determined $h_n \in H(n)$ such that

$$h_0 = h, \quad E^+ h_n = h_{n+2}.$$

Then we define the number $c_n \neq 0$ by

$$E^- h_n = c_{n-2} h_{n-2}.$$

The system of numbers $(c_n)_{n \text{ even}}$ is independent of the choice of h. It is clear that the action of $\mathfrak{g}_\mathbb{C}$ is determined by this system of numbers and it is also clear that isomorphic representations lead to the same system. A much better result is true. The relation $[E^+, E^-] = \mathrm{i}W$ shows

$$c_n - c_{n-2} = 4n.$$

Hence all c_n are determined by one, for example by c_0. We see that the isomorphy type of the representation is determined by one parameter! The case 2) is similar. We obtain the following result.

9.5 Proposition. *Consider an admissible representation of type 1). Then there exists a parameter c such that*

$$E^+ E^- h = ch \quad \textit{for} \quad h \in H(0)$$

The representation is determined up to isomorphism by c. In the case of type 2) the same result holds if one replaces $H(0)$ by $H(1)$.

Assume now that there is a highest weight n. Then we choose $h \in H(n)$. As in the previous case one shows that the parameter c defined by $E^+ E^- h = ch$ determines the representation. But now $E^+ h = 0$. Hence the relation $[E^+, E^-] = \mathrm{i}W$ gives a better result, namely

$$c = 4m.$$

9.6 Proposition. *An admissible representation with a highest weight vector is determined up to isomorphy by its highest weight n. If h is a highest weight vector then*

$$E^+ E^- h = 4nh.$$

The same is true for lowest weight representations if one replaces the equation by

$$E^- E^+ h = 4nh.$$

10. The Bargmann classification

Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representations. Then the derived representation $\mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(H(K))$ has the property that the operators $d\pi(A)$ for $A \in \mathfrak{g}$ are skew symmetric. Hence it is natural to ask for an admissible representation $\mathfrak{g}$ on $\mathcal{H}$ whether there exists a hermitian scalar product on $\mathcal{H}$ such that the elements of $\mathfrak{g}$ act skew-symmetrically. If this is case we call (by some misuse of language) the representation unitarizible.

10.1 Proposition. *Consider a unitarizible admissible representation. If it has no highest or lowest weight then its parameter c (s. Proposition 9.5) is real and negative.*

If it has a lowest weight n then $n < 0$. If it has a highest weight n then $n > 0$.

Corollary. *If there exists a lowest or highest weight then $H(0)$ is 0. So $H(0)$ is different form zero only in the case 1) of Proposition 9.4.*

So far we only derived properties of unitary representations but we do not know yet whether they exist.

10.2 Lemma. *For each real number $c < 0$ there exists an even and an odd admissible representation without highest or lowest weight with this parameter. Each integer $n > 0$ occurs as highest weight and each integer $n < 0$ occurs as lowest weight of an admissible representation*

Proof. We take for example the case of an even representation without highest or lowest weight. We take for each even n a one-dimensional vector space $H(n)$ and define $\mathcal{H}$ to be the algebraic sum of $H(n)$. Then we define the operators W, E^+, E^- by the necessary formulae which we derived above. It is rather clear that this gives an admissible representation. The definition of the scalar product is also clear. The other cases are similar. □

The question arises whether each admissible representation can be realized by unitary representations of G (in the sense that it is isomorphic to its derived representation).

10.3 Theorem. *Each unitarizible admissible representation can be realized by an irreducible unitary representation.*

The proof will be given in the next section by means of explicit constructions in each of the cases. □

We recall that an irreducible unitary representation is determined up to unitary equivalence by the isomorphy type of the underlying admissible representation.

Collecting together, we get the classification of the irreducible unitary representations of G. First we introduce the usual notations. Recall that in case 1) and case 2) the representation is determined by a single parameter $c < 0$ which can be arbitrary in both cases. Instead of c we will use a new parameter s.

In case 1) (even case, no highest or lowest weight) we define s by

$$(s+1)(s-1) = c.$$

The number $(s+1)(s-1)$ is a negative real number if and only if

1) s is purely imaginary ($c \le -1$)
2) s is real and $-1 < s < 1$. ($-1 < c < 0$)

We notice that the solution s is unique up to the replacement $s \mapsto -s$. We use the following notation:

The even principal series consists of all representations of even type without highest or lowest weight and with the property that s is purely imaginary.

The complementary series consists of all representations of even type without highest or lowest weight and with the property that $s \in (-1, 1)$ but $s \neq 0$.

Next we treat case 2) (odd case, no highest or lowest weight). In this case we introduce s by

$$s^2 = c.$$

Hence s is different from zero and purely imaginary. It is determined up to sign.

The odd principal series consists of all representations of odd type without highest or lowest weight and with the property that s is different from zero and purely imaginary.

In the case 3) (there exists a highest weight n) we define

$$s = n - 1.$$

This is an integer ≥ 0 and finally in the case 4) (there exists a lowest weight n) we define

$$s = n + 1.$$

This is an integer ≥ 0.

The representation with highest weight 1 or lowest weight -1 (in both cases $s = 0$) have some special properties. Hence they are separated from the other representations with a highest or lowest weight vector. Those with $s \neq 0$ define the *discrete series* and the two with $s = 0$ define the mock discrete series.

Collecting together we obtain Bargmann's classification.

10.4 Theorem. *Each unitary irreducible unitary representation of $G = \mathrm{SL}(2,\mathbb{R})$ is unitary isomorphic to a representation of the following list.*

1) *The even principal series, $s \in \mathrm{i}\mathbb{R}$,*
2) *the odd principal series, $s \in \mathrm{i}\mathbb{R} - \{0\}$,*
3) *the complementary series, $s \in (-1,1) - \{0\}$,*
4) *the discrete series, $s \in \mathbb{Z} - \{0\}$,*
5) *the mock discrete series $s = 0$ (two representations).*

In the first three cases, s is determined up to its sign. In the last two cases s is uniquely determined.

We mention that $s = 0$ occurs three times in this list (the member $s = 0$ of the even principal series and the two mock discrete series). The parameters in $\mathrm{i}\mathbb{R} - \{0\}$ occur twice (in the two principal series). The values $s \in \mathbb{Z} - \{0\}$ occur only once.

Why has the mock discrete series been separated from the discrete series? If G is an arbitrary locally compact group, one has a general notion of a discrete series representation. An irreducible unitary representation is called a discrete series representation of it occurs (as unitary representation) in the regular representation $L^2(G)$. It can be shown that the discrete series representations of $G = \mathrm{SL}(2,\mathbb{R})$ in this sense consist of all representations with a higher or lower weight vector with two exceptions, the weights 1 and -1 do not occur. Hence these play a special role. Since they look similar as the discrete series representations they are called “mock discrete”.

11. Induced representations

The basic idea of induced representations is easy to explain in the case of finite groups. Let $P \subset G$ be a subgroup of a finite group and $\sigma : P \to \mathrm{GL}(H)$ a representation of the subgroup. We consider the space $\mathrm{Ind}(\sigma)$ of all functions $f : G \to H$ with the property

$$f(px) = \sigma(p)f(x) \quad \text{for} \quad p \in P,\ x \in G.$$

Then G acts by right translation on $\mathrm{Ind}(\sigma)$. This simple principle carries over to locally compact groups with some modifications. One would like that the induced representation of a unitary representation is unitary. This causes some problem. One problem is that in important cases the condition $\Delta_G|\Delta_P$ may be false so there is no G-invariant measure on $P\backslash G$.

Let us assume in a first step that both G and P are unimodular (and that σ is a unitary representation. Let $f : G \to V$ be a function with the property

$$f(px) = \sigma(p)f(x).$$

The function $\|f(x)\|$ is left-invariant under P and can be considered as function on $P\backslash G$. Then we consider the space of all measurable functions $f : G \to H$ such that $\|f(x)\|^2$ is integrable considered as function on $P\backslash G$. This space carries the scalar product

$$\langle f, g\rangle = \int_{P\backslash G} \langle f(x), g(x)\rangle dx.$$

(The scalar produce is also P-invariant.) Now we define the Hilbert space

$$\mathrm{Ind}(\sigma) = \mathrm{Ind}_P^G(\sigma)$$

as the quotient of this space by the space of all functions which have the property that $\|f(x)\|$ vanishes outside a zero set of $P\backslash G$. This gives a unitary representation of G.

Unfortunately this construction is not good enough. For example in the case $G = \mathrm{SL}(2, \mathbb{R})$ and P is the subgroup of upper triangular matrices this condition is not satisfied. The group G is unimodular but P not. The following procedure to overcome this difficulty is due to Mackey. We make a very weak assumption.

11.1 Assumption. *There exists a positive continuous function $q : G \to \mathbb{R}_{>0}$ with the property*

$$q(px) = \frac{\Delta_P(p)}{\Delta_G(p)} q(x).$$

In all cases that we will meet this condition will be satisfied. But it is not really necessary. It is sufficient to have a measurable q with this property. But we do not want to talk to much about measurability.

There is an important case where the existence of a function q is trivial.

11.2 Remark. *Let $P, K \subset G$ we closed subgroups of a locally compact group such that the map*

$$P \times K \xrightarrow{\sim} G, \quad (p, k) \longmapsto pk,$$

is topological. Then the function

$$q(pk) = \frac{\Delta_P(p)}{\Delta_G(p)}$$

satisfies the Assumption 11.1.

An example is $G = \mathrm{SL}(2, \mathbb{R})$, P group of upper triangular matrices, $K = \mathrm{SO}(2, \mathbb{R})$. Note that in this case $\Delta_G|P$ is different from Δ_P, since G is unimodular but P not.

We need a generalization of the construction of quotient measures (Proposition I.3.3).

11.3 Proposition. *Assume that $P \subset G$ is a closed subgroup of a locally compact group and that q is a function as in Assumption 11.1. Then there exists a unique Radon measure $d\bar{x}$ on $P\backslash G$ (depending on q) such that the formula*

$$\int_G f(x)q(x)d_r x = \int_{P\backslash G} \left[\int_P f(px)d_r p \right] d\bar{x}$$

holds for $f \in \mathcal{C}_c(G)$. Here $d_r p$ denotes a right invariant measure on P. and $d_r x$ a left invariant measure on G (suitably normalized).

Notice that the inner integral is left invariant as function of x, since $d_r p$ has been taken to be right invariant.

Usually we will write dx instead of $d\bar{x}$ as long this is not expected to cause confusion.

This measure on $P\backslash G$ is not invariant under the action of G. But it has still the weaker problem that the space of zero functions is invariant under (right) translation with elements of G.

One can use this measure to define the induced representation of a unitary representation $\sigma : P \to \mathrm{GL}(H)$.

11.4 Definition and Remark. *Assume that $P \subset G$ is a closed subgroup of a locally compact group and that q is a function as in Assumption 11.1. Let dx be the corresponding measure on $P\backslash G$. Let $\sigma : P \to \mathrm{GL}(H)$ be a unitary representation. Consider the space of all measurable functions $f : G \to H$ with the property $f(px) = \sigma(p)f(x)$ and such that $\|f(x)\|_\sigma^2$ is integrable considered as function on $P\backslash G$. The quotient of this space by the subspace of all functions, such that $\|f(x)\|_\sigma^2$ is a zero function (considered on $P\backslash G$), is a Hilbert space $H(\sigma)$ with the hermitian inner product*

$$\langle f, g \rangle = \int_{P\backslash G} \langle f(x), g(x) \rangle_\sigma dx.$$

The group G acts on it by means of the modified translation from the right: for $g \in G$ the operator R_g is defined by

$$(R_g f)(x) = f(xg)\sqrt{\frac{q(x)}{q(xg)}}.$$

This is a unitary representation, called the (unitary) induced representation of σ to G. It is independent of the choice of q up to unitary isomorphism.

12. The principal and the mock discrete series

We consider the Iwasa decomposition $G = PK = ANK$ of $G = \mathrm{SL}(2,\mathbb{R})$. We fix a complex number s. For any function f we consider

$$F(kan) = \varrho(a)^s f(k) \quad \text{where} \quad \varrho\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a.$$

This gives an identification between functions on K and functions on G with the property

$$F(anx) = \varrho(a)^s F(x).$$

The function f is measurable if and only if F is measurable. We denote the space of all measurable functions F such that f is square integrable by $\mathcal{H}(s)$ any by $H(s)$ the quotient by the subspace of all such F such that f is a zero function. Then we have natural isomorphisms

$$\mathcal{H}(s) \cong \mathcal{L}^2(K, dk), \quad H(s) \cong L^2(K, dk).$$

The group G acts on the space $\mathcal{H}(s)$ by translation form the right. It is clear that this factors through $H(s)$.

12.1 Remark. *Let s a complex number. Translation from the right defines a continuous representation π_s of $G = \mathrm{SL}(2,\mathbb{R})$ on $H(s)$ (which can also be considered as representation on $L^2(K)$). This representation is unitary if s is purely imaginary.*

In the unitary case this is just a special case of the induce representation introduced in the previous section (Proposition 11.3). One takes in this context the function q described in Remark 11.2. In our case it is $q(ank) = \varrho(a)^2$.

12.2 Lemma. *Let $f \in \mathcal{H}(s)$ be an element that is $\mathcal{C}^\infty$ considered as function on the group G. Then f is a $\mathcal{C}^\infty$ vector of the representation π_s and we have*

$$\mathcal{L}_X f = d\pi_s(X) f.$$

The space $H(n,s)$ of all K-eigenfunctions which pick up the nth power of the standard character is one dimensional and generated by the function

$$\varphi\left(\left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\left(\cos\theta \quad \sin\theta \quad -\sin\theta \quad \cos\theta\right)\right) = y^{s+1}e^{\mathrm{i}n\theta}.$$

We can use the formula in Proposition 6.1 to compute the derived representation. The result is

$$\boxed{\begin{aligned} d\pi_s(W)\varphi_n &= \mathrm{i}n\varphi_n, \\ d\pi_s(E^-)\varphi_n &= (s+1-n)\varphi_{n-2} \\ d\pi_s(E^+)\varphi_n &= (s+1+n)\varphi_{n+2} \end{aligned}}$$

From this description, it is easy to to invariant subspaces, namely

$$H(s)^{\mathrm{even}} = \widehat{\bigoplus_{n \text{ even}}} \mathbb{C}\varphi_n, \quad H(s)^{\mathrm{odd}} = \widehat{\bigoplus_{n \text{ odd}}} \mathbb{C}\varphi_n.$$

If s is not an integer both representations are irreducible. In the case that s is purely imaginary we get unitary representation. In this way we get realizations of the two principal series where $s = 0$ has been excluded.

In the case $s = 0$ the odd space can be decomposed into subspaces again. Hence we obtain in the case $s = 0$ three irreducible subspaces

$$\widehat{\bigoplus_{n \text{ even}}} \mathbb{C}\varphi_n, \quad \widehat{\bigoplus_{n \geq 1 \text{ odd}}} \mathbb{C}\varphi_n, \widehat{\bigoplus_{n \leq 1 \text{ odd}}} \mathbb{C}\varphi_n.$$

Hence we have found realizations of the principal series and the tow mock discrete representations.

One is tempted to try to construct also the complementary and the discrete series in this way. For example the complementary series seems to be related to $H(s)^{\mathrm{odd}}$ for $s \in (-1, 1)$, $s \neq 0$. The problem is that for these s the representation π_s is not unitary. So there is a problem to get a unitary realization along these lines.

13. The discrete series

We denote by $\mathbb{H}$ the upper half plane in the complex plane. Recall that the group $G = \mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}$ through $(az + b)(cz + d)^{-1}$. The measure $dxdy/y^2$ is invariant under the action of G. We consider more generally the measures

$$d\omega_n = y^n \frac{dxdy}{y^2}.$$

Then we consider the space

$$H = L^2(\mathcal{H}, \omega_n)$$

of all *holomorphic* functions which are square integrable with respect to this measure,

Chapter III. Representations of the Poincaré group

1. The Lorentz group

The Minkowski space of dimension $n+1$ is the vector space $\mathbb{R}^{n+1}$ that has been equipped with the symmetric bilinear form

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots x_{n+1}y_{n+1}.$$

A vector is called time-like if $\langle x, x\rangle = 0$. The set of all time like vectors consists of two connected cones. One of them is define by $x_1 > 0$. We call this the future cone.

The Lorentz group is the subgroup of $\mathrm{GL}(\mathbb{R}^{n+1})$ that preserves this form, $\langle gx, gy\rangle = \langle x, y\rangle$. If one identifies $\mathrm{GL}(\mathbb{R}^{n+1})$ with $\mathrm{GL}(n+1, \mathbb{R})$ in the usual manner, then this means

$$A'JA = J \quad \text{where} \quad J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We denote the Lorentz group by $\mathrm{O}(n, 1)$. We always assume $n > 0$. There are two important subgroups. The first is the special orthogonal group $\mathrm{SO}(n, 1)$ which consists of all elements with determinant one. The second is the subgroup $\mathrm{O}^+(n, 1)$ that preserves the future cone. Since time like vectors are mapped to time like vectors, it is sufficient to know that the vector $(1, 0, \ldots, 0)$ is mapped to a vector a with $a_1 \geq 0$. For the matrix A this means that $a_{11} > 0$. Hence we have seen that the set of all matrices in the Lorentz group with this property build a group. The elements of this group are called loxodromic. The matrix J is in the Lorentz group and has determinant -1. This shows

$$\mathrm{O}(n, 1) = \mathrm{SO}(n, 1) \cup \mathrm{SO}(n, 1)J.$$

The negative of the unit matrix E is not loxodromic. Hence we see

$$\mathrm{O}(n,1) = \mathrm{O}^+(n,1) \cup \mathrm{O}^+(n,1)(-E).$$

We use the notation

$$\mathrm{SO}^+(n,1) = \mathrm{O}^+(n,1) \cap \mathrm{SO}(n,1).$$

We see

$$\mathrm{O}(n,1) = \mathrm{SO}^+(n,1) \ \cup\ \mathrm{SO}^+(n,1)J \ \cup\ \mathrm{SO}^+(n,1)(-E) \ \cup\ \mathrm{SO}^+(n,1)(-J).$$

It can be shown that $\mathrm{SO}^+(n,1)$ is open in $\mathrm{O}(n,1)$ and connected. Hence $\mathrm{O}(n,1)$ has 4 connected components.

For small n one can find different descriptions. We start with $\mathrm{O}(2,1)$. For this we consider the vector space $\mathcal{X}$ of all skew symmetric real 2×2-matrices

$$X = \begin{pmatrix} x_2 & x_1 \\ -x_1 & x_3 \end{pmatrix}.$$

Their determinant is $-x_1^2 + x_2^2 + x_3^2$. We identify $\mathcal{X}$ with $\mathbb{R}^3$ in the obvious way. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{X}$ through $(A,X) \longmapsto AXA'$. For given A this transformation can be considered as element of $\mathrm{GL}(3,\mathbb{R})$. The above formula for the determinant shows that it is in $\mathrm{O}(3,\mathbb{R})$. From the Iwasawa decomposition one can see that $\mathrm{SL}(2,\mathbb{R})$ is connected. Hence we constructed a homomorphism $\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(3,1)$.

1.1 Proposition. *The homomorphism*

$$\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(3,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(3,1)$ *has precisely two inverse images which differ only by the sign.*

We skip the proof of the surjectivity. □

Proposition 1.1 is only a special case of a general result. For each n there exists connected locally compact group G and a continuous surjective homomorphism $G \to \mathrm{SO}^+(n,1)$ such that each element of the image has precisely two pre-images. This group is (in an obvious sense) essentially unique and called the spin covering. The usual notation is $\mathrm{Spin}(n,1)$ for this group. So $\mathrm{Spin}(2,1) = \mathrm{SL}(2,\mathbb{C})$ We don't give this (not quite trivial construction) in the general case and treat only the case $n=3$ which is fundamental for physics.

For the construction of $\mathrm{Spin}(3,1)$ ce consider the space of all hermitian 2×2-matrices

$$H = \begin{pmatrix} h_0 & h_1 \\ \bar h_1 & h_2 \end{pmatrix}.$$

We identify $\mathcal{H}$ with $\mathbb{R}^4$ through

$$H \longmapsto \Big(\frac{h_0 + h_2}{2}, \frac{h_0 - h_2}{2}, \operatorname{Re} h_1, \operatorname{Im} h_1\Big).$$

Then we have

$$\det H = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{H}$ through $(A,H) \longmapsto AH\bar{H}'$ It preserves the determinant. Hence we obtain a Lorentz transformation. It can be shown that $\mathrm{SL}(2,\mathbb{C})$ is connected two. Hence we get a homomorphism

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(1,3).$$

1.2 Proposition. *The homomorphism*

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(3,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(3,1)$ *has precisely two inverse images which differ only by the sign.*

This allows us to write $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$.

The existence of spin coverings is not tied to signature $(n,1)$. For example we can consider the Euclidian orthogonal group $\mathrm{O}(3,\mathbb{R})$. Recall that $\mathrm{O}(n,\mathbb{R})$ consists of all $A \in \mathrm{GL}(n,\mathbb{R})$ with the property $A'A = E$. This is a closed subgroup. The rows and columns have Euclidean length 1. Hence $\mathrm{O}(n,\mathbb{R})$ is a compact group (in contrast to the Lorentz group!). The subgroup $\mathrm{SO}(n,\mathbb{R})$ of elements of determinant one is called the special orthogonal group. One can show that it is connected. The group $\mathrm{O}(n,\mathbb{R})$ can be embedded into the Lorentz group $O(n,1)$ by means of

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

We consider this in the case $n = 3$. We can consider the inverse image in $\mathrm{SL}(2,\mathbb{C})$. One can check that this inverse image is the special unitary group $\mathrm{SU}(2)$. Recall that The unitary group $\mathrm{U}(n)$ is the subgroup of all $A \in \mathrm{GL}(n,\mathbb{C})$ with the property $\bar{A}'A = E$. This is a compact group. The special unitary group is the subgroup of all A with $\det A = 1$. One can show that it is connected.

1.3 Proposition. *The homomorphism*

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3,\mathbb{R})$$

is continuous and surjective. Each element of $\mathrm{SO}(3,\mathbb{R})$ *has precisely two inverse images which differ only by the sign.*

Hence we can write $\mathrm{Spin}(3,\mathbb{R}) = \mathrm{SU}(2,\mathbb{C})$.

2. The Poincaré group

In the following we call $\mathrm{O}(n,1)$ the *homogeneous Lorentz group.* The *inhomogeneous Lorentz group* is the set of all transformations of $\mathbb{R}^{n+1}$ of the form

$$v \longmapsto A(v) + b$$

where A is a Lorentz transformation and $b \in \mathbb{R}^{n+1}$. This group can be identified with the set $\mathrm{O}(n,1) \times \mathbb{R}^{n+1}$. The group law then is

$$(g,a)(h,b) = (gh, a + gb).$$

We write for the inhomogeneous Lorentz group simply

$$\mathrm{O}(n,1)\mathbb{R}^{n+1}.$$

We want to define also a "spin covering". For this we have to consider the action of $\mathrm{Spin}(n,1)$ on $\mathbb{R}^{n+1}$ which is defined by means of the natural homomorphism $\mathrm{Spin}(n,1) \to \mathrm{O}(n,1)$ and the natural action of $\mathrm{O}(n,1)$ on $\mathbb{R}^{n+1}$. We write this action simply in the form $(g,v) \mapsto gv$. The *Poincaré group* $P(n)$ is the set

$$P(n) = \mathrm{Spin}(n,1) \times \mathbb{R}^{n+1}$$

together with the group law

$$(g,a)(h,b) = (gh, a + gb).$$

It is clear that this is a group and that the natural map

$$P(n) \longrightarrow \mathrm{O}(n,1)\mathbb{R}^{n+1}$$

(spin covering on the first factor and identity on the second factor) is a homomorphism. This image is $\mathrm{SO}^+(n,1)\mathbb{R}^{n+1}$ and each element has two inverse images.