Riemann Surfaces

Lecture 2007/2008

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Inhalt

Chapter I. Topological spaces

1. The notion of a topological space

A topology $\mathcal{T}$ on a set X is a system of subsets —called open subsets— with the following properties:

1) $\emptyset$, X are open.
2) The intersection of finitely many open subsets is open.
3) The union of arbitrarily many open sets is open.

A *topological space* is pair $(X, \mathcal{T})$, consisting of a set X and a topology $\mathcal{T}$ on X. Usually we will write X instead of $(X, \mathcal{T})$, when it is clear which topology is considered. We give some constructions for topological spaces:

Metric spaces

Let (X, d) be a metric space. We denote by $U_r(a)$ the ball around a of radius r. A subset U of X is called open, if for every $a \in X$ there exists $\varepsilon > 0$ with the property $U_\varepsilon(a) \subset U$.

The induced topolgy

Let Y be a subset of a topological space $X = (X, \mathcal{T})$. Then Y can be equipped with the *induced topology* $\mathcal{T}|Y$.

A subset $V \subset Y$ belongs to $\mathcal{T}|Y$, if and only if there exists a subset $U \subset X$, $U \in \mathcal{T}$, such that

$$V = U \cap Y.$$

(When Y is an open subset of X then this means $V \in \mathcal{T}$.) Since a subset $V \subset Y$ is also a subset of X, one has to make clear whether "open" means to by in $\mathcal{T}$ or in $\mathcal{T}|Y$. If we say "open in X" we mean that V is contained in $\mathcal{T}$. Similarly "open in Y" means that it is contained in $\mathcal{T}|Y$.

The quotient topology

Let X be a topological space and $f : X \to Y$ a surjective map onto a set Y. Then Y can be equipped with the quotient topology: A subset $V \subset Y$ is called open if and only if the inverse image $U := f^{-1}(V)$ is open in X. There is an important special case: Let „$\sim$“ be an equivalence relation on X and let be $Y = X/\sim$ the set of equivalence classes and $f : X \to Y$ the canonical projection. Then Y is called the quotient space of X. Examples are a torus $X = \mathbb{C}/L$ for a lattice L or the modular space $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$.

The product topology

Let $X_1, \dots, X_n$ be a finite system of topological spaces. Then a natural topology on the cartesian product

$$X = X_1 \times \cdots \times X_n$$

can be defined:

A subset $U \subset X$ is called open if and only if for every point $a \in U$ there exist open subsets $U_1 \subset X_1, \dots, U_n \subset X_n$, such that

$$a \in U_1 \times \cdots \times U_n.$$

2. Continuous maps

A subset of a topological space $A \subset X$ is called *closed*, if the complement $X - A$ is open.

A subset $M \subset X$ is called a *neighborhood* of a point $a \in X$, if there exists an open subset $U \subset X$ with $a \in U \subset M$.

A point $a \in X$ is called a *boundary point* of a subset $M \subset X$, if every neighborhood of a contains points of M and of its complement $X - M$.

Notation.

$$\partial M := \text{set of boundary points},$$
$$\bar{M} := M \cup \partial M.$$

One shows easily that $\bar{M}$ is the smallest closed subset of X which contains M. Especially

$$M \text{ closed} \iff M = \bar{M}.$$

We call $\bar{M}$ the *closure* of M.

A map $f : X \to Y$ between topological spaces is called *continuous* in a point $a \in X$, if the inverse image $f^{-1}(V(b))$ of an arbitrary neighborhood of $b := f(a)$ is a neighborhood of a.

The following conditions are equivalent

1) The map f is continuous (i.e. continuous in every point).
2) The inverse image of an arbitrary open subset of Y is open in X.
3) The inverse image of an arbitrary closed subset of Y is closed in X.

The composition of two continuous maps is continuous. (This is true already in the pointwise sense.)

Some universal properties

Let Y be a subset of a topological space equipped with the induced topology. A map $f : Z \to Y$ of a third topological space into Y is continuous if and only if the composition with the natural inclusion $Y \hookrightarrow X$ is a continuous map $Z \to X$.

Let $f : X \to Y$ be a surjective map of topological spaces, where Y carries the quotient topology. A map $Y \to Z$ into a third topological space Z is continuous if and only if the composition with f is a continuous map $X \to Z$.

Let $X_1, \dots, X_n$ be topological spaces and

$$f : Y \longrightarrow X_1 \times \cdots \times X_n$$

a map of another topological space Y into the cartesian product (equipped with the product topology). The map f is continuous if and only if each component

$$f_j : Y \longrightarrow X_j \qquad (f = (f_1, \dots, f_n))$$

is continuous.

Topological maps

A map $f : X \to Y$ between topological spaces is called *topological*, if it is bijective and if f und f^{-1} both are continuous. Two topological spaces X, Y are called *topologically equivalent* or *homeomorphic* if there exists a topological map between them.

For example the 2-sphere

$$S^2 = \{\, x \in \mathbb{R}^3; \quad x_1^2 + x_2^2 + x_3^2 = 1 \,\}$$

and the *Riemann sphere* are homeomorphic. Here the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is equipped with the following topology. A subset $U \subset \bar{\mathbb{C}}$ is open if the following two conditions are satisfied:

a) $U \cap \mathbb{C}$ is open.
b) If $\infty \in U$ then there exists $C > 0$ such that $\{z \in \mathbb{C};\ |z| > C\} \subset U$.

The topological map $S^2 \to \bar{\mathbb{C}}$ is constructed by means of the stereographic projection.

If $L \subset \mathbb{C}$ is a lattice, then $\mathbb{C}/L$ is homeomorphic to the cartesian product of two circles:

$$\boxed{\mathbb{C}/L = \text{Torus} \simeq S^1 \times S^1.}$$

3. Special classes of topological spaces

A topological space X is called *Hausdorff* if for two different points $a, b \in X$ there exist disjoint neighborhoods $U(a)$ and $U(b)$.

A topological space X ic called *compact*, if it is Hausdorff and if every open covering admits a finite sub-covering. A subset Y of a topological space X is called compact, if it is —equipped with the induced topology— a compact topological space.

Properties of compact subsets

a) Compact subsets are closed.
b) A closed subset of a compact space is compact.
c) Let $f : X \to Y$ be a continuous map of Hausdorff spaces, then the image of a compact subset is compact.
d) Let X be a compact and Y a Hausdorff space and $f : X \to Y$ bijective and continuous. Then f is topological.
e) The product $X_1 \times \cdots \times X_n$ of compact spaces is compact.

Locally compact spaces and proper maps

A topological space is called *locally compact* if it is Hausdorff and if every point admits a compact neighborhood.

A continuous map $f : X \longrightarrow Y$ of locally compact spaces is called *proper* if the inverse image of an arbitrary compact subset is compact.

We formulate two basic properties of proper maps.

3.1 Remark. *Let $f : X \to Y$ be a proper map. The image $f(A)$ of a closed subset $A \subset X$ is closed.* BiEiAb

Use that a subset of a locally compact space is closed if and only if its intersection with compact subsets is compact. □

3.2 Remark. *Let $f : X \to Y$ be a proper map and let $K \subset Y$ be a compact subset. For every open subset $U \subset X$ which contains $f^{-1}(K)$ there exists an open subset $K \subset V \subset Y$ with the property* EigFas

$$f^{-1}(V) \subset U.$$

Take for V the complement of $f(X - U)$. □

4. Sequences

A sequence (a_n) in a Hausdorff space X converges to $a \in X$ if for every neighborhood $U(a)$ there exists $N \in \mathbb{N}$ with $a_n \in U$ for $n \geq N$. The limit a is unique. If $f : X \to Y$ is continuous in a then $a_n \to a$ implies $f(a_n) \to f(a)$.

It is easy to show that in a compact space every sequence admits a convergent sub-sequence. The converse is only true under additional conditions.

One says that a topological space has a *countable* basis of the topology if there exists a countable family (U_i) of open subsets such that every open subset can be written as union of members of this system. If this is the case, also every subset equipped with the induced topology has countable basis of the topology. The space $\mathbb{R}^n$ with the usual topology has countable basis. One can take open balls where the radius and the coordinates of the center are rational numbers.

4.1 Proposition. *Assume that X is a Hausdorff space such that every sequence admits a convergent sub-sequence. Then X is compact if one of the following two conditions is satisfied:* Hfk

a) *The topology comes from a metric.*
b) *There exists a countable basis of the topology.*

5. Connectedness

A topological space is called *arc-wise connected* if every two points are contained in (the image of) an arc. An arc is a continuous map of a real interval into X.

A topological space is called *connected* if one of the following two equivalent conditions is satisfied:

1) Every locally constant map $f : X \to M$ into an arbitrary set M is constant. It is sufficient to verify this for one set M which contains at least two elements.
2) If $X = U \cup V$ is written as union of two disjoint open subsets U, V, then one them is empty.

The mean value theorem of calculus shows that every real interval is connected. As a consequence every arcwise connected space is connected. The converse is only true under additional assumptions (s. below).

arc components

Two points of a topological space are called equivalent if there exists an arc which contains both. The equivalence classes of this equivalence relation are called *arc components*.

A *topological manifold* is a Hausdorff space such that every point admits an open neighborhood which is homeomorphic to some open subset of $\mathbb{R}^n$. If n can be taken to be two, then X is called a *(topological) surface*.

5.1 Remark. *Let X be a topological manifold. Then the arc components are open. The manifold is connected if and only if it is arcwise connected.* BzZ

Since an open subset of a manifold is a manifold, the arc components of a manifold are also manifolds. Of course they are connected. We call the arc components of a manifold also the *connected components*.

6. Paracompactness

A covering $\mathfrak{U} = (U_i)_{i\in I}$ of a topological space is called *locally finite,* if for every point $a \in X$ there exists a neighborhood W, such that the set of indices $i \in I$ with $U_i \cap W \neq \emptyset$ is finite.

A covering $\mathfrak{V} = (V_j)_{j\in J}$ is called a *refinement* of the covering $\mathfrak{U}$ if for every index $j \in J$ there exists an index $i \in I$ with $V_j \subset U_i$. If one chooses for each j such an i one obtains a so-called *refinement map* $J \to I$, which needs not to be unique.

6.1 Definition. *A Hausdorff space is called* ***paracompact*** *if every open covering admits a locally finite (open) refinement.* DPc

We collect some results about paracompact spaces without proofs. Firstly we give examples:

Every metric space is paracompact.

Every locally compact space with countable basis of topology is paracompact.

Next we formulate the basic result about paracompactness: Let $\mathfrak{U} = (U_i)$ be a locally finite covering. A partition of unity with respect to $\mathfrak{U}$ is family φ_i of continuous real valued functions on X with the following property:

a) The support of φ_i is compact and contained in U_i.
b) $0 \le \varphi_i \le 1$,
c) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in X$.

(This sum is finite.)

6.2 Proposition. *Let X be a paracompact space. For every locally finite open covering there exists a partition of unity.* EPu

We mention two related results:

6.3 Proposition. *Let X be a paracompact space and $\mathfrak{U} = (U_i)$ a locally finite open covering. There exist open subsets $V_i \subset U_i$ whose closure $\bar{V}_i$ (taken in X) is contained in U_i and such that $\mathfrak{V} = (V_i)$ is still a covering.* VeU

Another related result states:

6.4 Proposition. *Let X be a locally compact paracompact space, U an open subset and $V \subset\subset U$ a relatively compact open subset in U. Then there exists a continuous function on X which is one on V and whose support is compact and contained in U.* BF

(The symbol $V \subset\subset U$ means that the closure $\bar{V}$, taken in X, is compact and contained in U.)

7. Frèchet spaces

A semi-norm p on a complex vector space V is a map $p : V \to \mathbb{R}$ with the properties

a) $p(a) \geq 0$ for all $a \in V$,
b) $p(ta) = |t|p(a)$ for all $t \in \mathbb{C}$, $a \in V$,
c) $p(a + b) \leq p(a) + p(b)$.

The ball of radius $r > 0$ is defined as

$$U_r(a, p) := \{\, x \in V;\ p(a - x) < r \,\}.$$

Let $\mathcal{M}$ be a set of semi-norms. A subset $B \subset V$ is called a semi-ball around a with respect to $\mathcal{M}$ if there exists a finite subset $\mathcal{N} \subset \mathcal{M}$ and a $r > 0$ such that

$$B = \bigcap_{p \in \mathcal{N}} U_r(a, p).$$

A subset U of V is called open (with respect to $\mathcal{M}$) if for every $a \in U$ there exists a semi-ball B around a with $B \subset U$.

It is clear that this defines a topology on V such that all $p : V \to \mathbb{C}$ are continuous. (It is actually the weakest topology with this property.) It is also easy to to see that the maps

$$V \times V \longrightarrow V, \quad (a, b) \longmapsto a + b, \qquad \mathbb{C} \times V \longrightarrow V, \quad (t, a) \longmapsto ta,$$

are continuous. (This means that V is a topological vector space.) Moreover a sequence (a_n) in V converges to $a \in V$ if and only if $p(a_n - a) \to 0$ for all $p \in \mathcal{M}$.

The set $\mathcal{M}$ is called definit, if

$$p(a) = 0 \quad \text{for all} \quad p \in \mathcal{M} \quad \Longrightarrow \quad a = 0.$$

It is easy to prove that X is a Hausdorff space for definit $\mathcal{M}$.

A sequence (a_n) in V is called a *Cauchy sequence* with respect to $\mathcal{M}$, if for every $\varepsilon > 0$ and every $p \in \mathcal{M}$ there exists an $N = N(p, \varepsilon)$ such that

$$p(a_n - a_m) < \varepsilon \quad \text{for} \quad n, m \geq N.$$

The set $\mathcal{M}$ is called of countable type, if there exists a countable subset $\mathcal{N} \subset \mathcal{M}$ such that for every $p \in \mathcal{M}$ there exists a $q \in \mathcal{N}$ with the property $p \leq q$. Then $\mathcal{M}$ and $\mathcal{N}$ obviously define the same topology and the same Cauchy sequences.

7.1 Definition. *A Frèchet space $(V, \mathcal{M})$ is a pair consisting of a complex vector space and a set of semi-norms on V such the following properties are satisfied:* DFR

a) $\mathcal{M}$ is definit.
b) $\mathcal{M}$ is of countable type.
c) Every Cauchy sequence converges.

Notice that a Banach space is a Frèchet space, where $\mathcal{M}$ consist of a single element.

7.2 Lemma. *Frèchet spaces are metrizable.* Fsm

Proof. We choose some ordering of $\mathcal{N} = \{p_1, p_2, \ldots\}$. Then one defines

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(a - b)}{1 + p(a_n) + p(b_n)}.$$

It is easy to show that this is a metric which defines the original topology.

□

Basic example of Frèchet spaces

Let $U \subset \mathbb{C}$ be an open subset and $\mathcal{O}(U)$ the set of all analytic functions on U. This is a complex vector space. For an arbitrary compact subset $K \subset U$ we define

$$p(f) = p_K(f) := \max_{z \in K} |f(z)|.$$

This is s semi norm. A sequence (f_n) converges with respect to p_K if and only if f_n converges uniformly on K.

7.3 Remark. *Let $U \subset \mathbb{C}$ be an open subset. The vector space $\mathcal{O}(U)$ equipped with the set of all norms of the form p_K, $K \subset U$ compact, is a Frèchet space.* HFF

The set of all p_K is of countable type: It is enough to take for K closed discs with rational coordinates of the center and rational radius. The convergence of Cauchy sequences follows from the theorem of Weierstrass, which states that analyticity is stable under uniform convergence. □

The basic result about this Frèchet space is:

7.4 Theorem of Montel. *Let U be an open subset of $\mathbb{C}$ and $C > 0$ a positive constant. The set* SvM

$$\mathcal{O}(U, C) := \left\{ f \in \mathcal{O}(U); \quad |f(z)| \leq C \text{ for } z \in U \right\}$$

is compact in $\mathcal{O}(U)$.

For the proof one has to use the fact that a metric space is compact if every sequence admits a convergent subsequence. Hence the statement follows from the usual theorem of Montel which states that every sequence in $\mathcal{O}(U, C)$ admits a locally convergent sub-sequence. We notice that the analogue for real differentiable functions is false. The proof uses heavily the Cauchy integral.

Compact operators

A well-known fact is that in a Banach space of infinite dimension the closed ball $||a|| \leq 1$ is not compact. This result is also true for Frèchet spaces in the following form:

Assume that the Frèchet space admits a non-empty open subset with compact closure. Then it is of finite dimension.

We need a generalization of this result: A continuous linear map $f : E \to F$ between Frèchet spaces is a *compact operator,* if there exists a non-empty open subset of E such that the closure of its image is compact. It is clear that this is the case if $f(E)$ is of finite dimension.

A linear map $f : V \to W$ is called *nearly surjective* if $W/f(V)$ has finite dimension. This is automatically the case when W is finite dimensional.

7.5 Theorem of Schwartz. *Let $f : E \to F$ be a surjective continuous linear map between Frèchet spaces and let $g : E \to F$ be a compact operator. Then $f + g$ is nearly surjective.* ToS

If one applies Schwartz's theorem in the case $E = F$, $f = -\operatorname{id}$ and $g = \operatorname{id}$ on obtains:

7.6 Corollary. *When the identity operator* $\mathrm{id} : E \to E$ *of a Frèchet space is compact, then E is finite dimensional* ToSc

Chapter II. Some algebra

1. Abelian groups

We assume that the reader is familiar with the notion of an abelian group and homomorphism between abelian groups. If A is a subgroup of an abelian group B, then the factor group B/A is well defined. All what one needs usually is that there is a natural surjective homomorphism $f : B \to B/A$ with kernel A. Let $f : B \to X$ be a homomorphism into some abelian group. Then f factors through a homomorphism $B/A \to X$ if and only if the kernel of f contains A. That f factors means that there is a commutative diagramm

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \\ A/B & & \end{array}$$

Let $f : A \to B$ be a homomorphism of abelian groups. Then the image $f(A)$ is a subgroup of B. If there is no doubt which homomorphism f is considered, we allow the notation

$$B/A := B/f(A).$$

1.1 Lemma. *A commutative diagram* HI

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

induces homomorphisms

$$B/A \longrightarrow D/C, \quad C/A \longrightarrow D/B.$$

A (finite or infinite) sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \cdots$$

is called exact at B if

$$\operatorname{Kernel}(B \longrightarrow C) = \operatorname{Image}(A \longrightarrow B).$$

It is called exact if it is exact at every place. An exact sequence $A \to B \to C$ induces an injective homomorphism

$$B/A \hookrightarrow C\ .$$

The sequence $0 \to A \to B$ is exact if and only if $A \to B$ is injective. The sequence $A \to B \to 0$ is exact if and only of $A \to B$ is surjective. The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if $A \to B$ is injective and if the induced homomorphism $B/A \to C$ is an isomorphism. A sequence of this form is called a *short exact sequence.* Hence the typical short exact sequence is

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \qquad (A \subset B).$$

1.2 The five term lemma. *Let* FTL

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagram with exact lines and such that f_1, f_2 and f_4, f_5 are isomorphisms. Then f_3 is an isomorphism too.

The proof is easy and left to the reader. □

2. Some homological algebra

A complex $A^\bullet$ is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that the composition of two consecutive is 0, $d_n \circ d_{n-1} = 0$. Usually one omits indices at the d-s and writes simply $d = d_n$ and hence $d \circ d = 0$, which sometimes is written as $d^2 = 0$. The cohomology groups of $A^\bullet$ are defined as

$$H^n(A^\bullet) := \frac{\mathrm{Kernel}(A^n \to A^{n+1})}{\mathrm{Image}(A^{n-1} \to A^n)} \qquad (n \in \mathbb{Z}).$$

They vanish if and only if the complex is exact. Hence the cohomology groups measure the absence of exactness of a complex.

A homomorphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \end{array}$$

It is clear how to compose two complex homomorphisms $f^\bullet; A^\bullet \to B^\bullet$, $g^\bullet; B^\bullet \to C^\bullet$ to a complex homomorphism $g^\bullet \circ f^\bullet : A^\bullet \to C^\bullet$. A sequence of complex homomorphisms

$$\cdots \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow \cdots$$

is called exact, if all the induced sequences

$$\cdots \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow \cdots$$

are exact. There is also the notion of a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

Here 0 stands for the zero-complex ($0^n = 0$, $d^n = 0$ for all n).

A homomorphism of complexes $A^\bullet \to B^\bullet$ induces naturally homomorphisms

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups (use 1.1). These homomorphisms are compatible with the composition of complex-homomorphisms. A less obvious construction is as follows: Let

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0$$

be a short exact sequence of complexes. We construct a homomorphism

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Let $[c] \in H^n(C^\bullet)$ be represented by an element $c \in C^n$. Take a pre-image $b \in B^n$ and consider $\beta = db \in B^{n+1}$. Since β goes to $d(c) = 0$ in C^{n+1} there exists a pre-image $a \in A^{n+1}$. This goes to 0 in A^{n+2} (because A^{n+2} is imbedded in B^{n+2} and b goes to $d^2(b) = 0$ there). Hence a defines a cohomology class $[a] \in H^{n+1}(A^\bullet)$. It is easy to check that this class doesn't depend on the above choices.

2.1 Fundamental lemma of homological algebra. *Let* LeS

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. Then the long sequence

$$\cdots \to H^{n-1}(C^\bullet) \xrightarrow{\delta} H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \to \cdots$$

is exact.

We leave the details to the reader. □

There is a second lemma of homological algebra which we will need.

2.2 Lemma. *Let* LHa

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} & \longrightarrow & A^{03} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} & \longrightarrow & A^{13} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} & \longrightarrow & A^{23} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{30} & \longrightarrow & A^{31} & \longrightarrow & A^{32} & \longrightarrow & A^{33} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

be a commutative diagram where all lines and columns are exact besides the first column and the first row (those containing A^{00}). Then there is a natural isomorphism between the cohomology groups of the first row and the first column,

$$H^n(A^{\bullet,0}) \cong H^n(A^{0,\bullet})$$

For $n = 0$ this is understood as

$$\text{Kernel}(A^{00} \longrightarrow A^{01}) = \text{Kernel}(A^{00} \longrightarrow A^{10}).$$

The proof is given by "diagram chasing". We only give a hint how it works. Assume $n = 1$. Let $[a] \in H^1(A^{0,\bullet})$ be a cohomology class represented by an element $a \in A^{0,1}$. This element goes to 0 in $A^{0,2}$. As a consequence the image of a in $A^{1,1}$ goes to 0 in $A^{1,2}$. Hence this image comes from an element $\alpha \in A^{1,0}$. Clearly this element goes to zero in $A^{2,0}$ (since it goes to 0 in $A^{2,1}$.) Now α defines a cohomology class $[\alpha] \in H^1(A^{\bullet,0})$. There is some extra work to show that this map is well-defined. □

3. The tensor product

All rings which we consider are assumed to be commutative and with unit elements. Ring homomorphisms are assumed to map the unit element into the unit element. A module M over a ring A is an abelian group together with a map $A \times M \to M$, $(a, m) \longmapsto am$, such that the usual axioms of a vector space are satisfied including $1_A m = m$ for all $m \in M$. The notion of linear maps, kernel, image of a linear map are as in the case of vector spaces. But in contrast to the case of vector spaces, a module has usually no basis. A module which admits a basis is called free. A finitely generated free module is isomorphic to R^n.

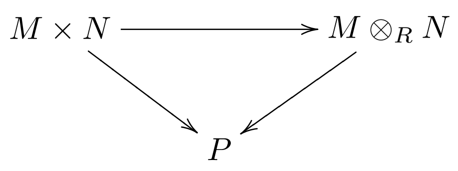
If $M \subset N$ is a submodule, then the factor group N/M carries a structure of an A-module. All what we have said about exact sequences of abelian groups is literarily true for A-modules.

Tensor product

Recall that for two modules M, N over a ring R, there exists a module $M \otimes_R N$ together with an R-bilinear map

$$M \times N \longrightarrow M \otimes_R N, \quad (a, b) \longmapsto a \otimes b,$$

such that for each bilinear map $M \times N \to P$ into an arbitrary third module P there exists a unique commutative diagram



with an R-linear map $M \otimes_R N \to P$. The tensor product $M \otimes_R N$ is generated by the special elements $m \otimes n$.

If $f : M \to M'$ and $g : N \to N'$ are R-linear maps, then one gets a natural R-linear map

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N', \quad (a, b) \longmapsto f(a) \otimes g(a).$$

It is clear that this map is uniquely determined by this formula. The existence follows from the universal property applied to the map $(a, b) \mapsto f(a) \otimes f(b)$.

Basic properties of the tensor product

There is a natural isomorphism

$$R \otimes_R M \xrightarrow{\sim} M, \quad (r \otimes m) \longmapsto rm$$

and more generally

$$R^n \otimes M \xrightarrow{\sim} M^n.$$

As a special case we get

$$R^n \otimes_R R^m \cong R^{n \times m}.$$

This is related also to the formula

$$(M \times N) \otimes_R P \cong (M \otimes_R P) \times (N \otimes_R P) \qquad \text{(canonically)}.$$

The tensor product is associative: For usual R-modules M, N, P on has an isomorphism

$$(M \otimes_R N) \otimes_R P \xrightarrow{\sim} M \otimes_R (N \otimes_R P), \quad (m \otimes n) \otimes p \longmapsto m \otimes (n \otimes p).$$

The existence of this map follows from the universal property of the tensor product.

The tensor product is also commutative:

$$M \otimes_R N \xrightarrow{\sim} N \otimes_R M, \quad m \otimes n \longmapsto n \otimes m.$$

Ring extension

Let $A \to B$ be a ring homomorphism and M an A-module. Then $M \otimes_A B$ carries a natural structure as B-module. It is given by $b(m \otimes b') = m \otimes (bb')$. The existence follows from the universal property of the tensor product. A special case is

$$A^n \otimes_A B = B^n.$$

Existence of the tensor product

For an arbitrary set I we define R^I to be the set of all maps $I \to R$, $i \mapsto r_i$ such that r_i is 0 for almost all i. So $R^I = R^n$ for $I = \{1, \ldots, \}$. By definition a module is free if and only if it is isomorphic to an R^I for suitable I. An arbitrary R-module M can be represented by an exact sequence

$$R^J \longrightarrow R^I \longrightarrow M \longrightarrow 0.$$

For another N module we define now the tensor product by the exact sequence

$$N^J \longrightarrow N^I \longrightarrow M \otimes_R N \longrightarrow 0.$$

The bilinear map $M \times N \to M \otimes_R N$ and the universal property are obvious.

Exactness properties

Let $M \to N$ be an injective homomorphism of R-modules. For an R-module P the induced homomorphism $M \otimes_R P \to N \otimes_R P$ needs not to be injective. But when $P \cong R^n$ is free, injectivity is preserved. A slight and trivial extension of this observation is:

3.1 Remark. *Let $M_1 \to M_2 \to M_3$ be an exact exact sequence of R-modules.* TbEx *Then for every* ***free*** *module P the sequence $M_1 \otimes_R P \to M_2 \otimes_R P \to M_3 \otimes_R P$ remains exact.*

In this connection we mention some other exactness properties. For two R-modules M, N we denote by $\mathrm{Hom}_R(M, M)$ the set of all R-linear maps $M \to N$. This is an R-module. Let $M \to N$ be an R-linear map. Then for an arbitrary R-module P one has obvious R-linear maps

$$\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N), \qquad \mathrm{Hom}_R(N, P) \longrightarrow \mathrm{Hom}_R(M, P).$$

Since $\mathrm{Hom}(R^n, M) \cong M^n$, one has:

3.2 Remark. *Let $0 \longrightarrow M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of* HbEx *R-modules and P also an R-module, then:*

a) *If P is free then*

$$0 \longrightarrow \mathrm{Hom}_R(P, M_1) \longrightarrow \mathrm{Hom}_R(P, M_2) \longrightarrow \mathrm{Hom}_R(P, M_3) \longrightarrow 0$$

remains exact.

b) *If M_3 is free than*

$$0 \longrightarrow \mathrm{Hom}_R(M_3, P) \longrightarrow \mathrm{Hom}_R(M_2, P) \longrightarrow \mathrm{Hom}_R(M_1, P) \longrightarrow 0$$

remains exact.

We comment shortly b). When M_3 is free one can chose a system of elments in M_2 whose images in M_3 define a basis. This system generates a submodule $M_3' \subset M_2$ which maps isomorphically to M_3. Now it is easy to see that M_2 is isomorphic to $M_1 \times M_3$ and the map $M_1 \to M_2$ corresponds to $m \mapsto (m, 0)$ and the map $M_2 \to M_3$ corresponds to $(m_1, m_3) \longmapsto m_3$. Now the exactness shold be clear.

Chapter III. Sheaves

1. Presheaves

1.1 Definition. *A presheaf F (of abelian groups) on a topological space X is a map which assigns to every open subset $U \subset X$ an abelian group $F(U)$ and to every pair U, V of open subsets with the property $V \subset U$ a homomorphism* DPG

$$r_V^U : F(U) \longrightarrow F(V)$$

such that for three open subsets U, V, W with the property $W \subset V \subset U$

$$r_W^U = r_W^V \circ r_V^U$$

holds:

Example: $F(U)$ is the set al continuous functions $f : U \to \mathbb{C}$ and $r_V^U(f) := f|V$ (restriction).

Many presheaves generalize this example. Hence the maps r_V^U are called "restrictions" in general and one uses the notation

$$s|V := r_V^U(s) \qquad \text{for} \quad s \in F(U).$$

The elements of $F(U)$ sometimes are called "sections" of F over U. In the special case $U = X$ they are called "global" sections.

1.2 Definition. *Let X be a topological space. A homomorphism of presheaves* DAP

$$f : F \longrightarrow G$$

is a family of group homomorphisms

$$f_U : F(U) \longrightarrow G(U),$$

such that the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes for every pair $V \subset U$ of open subsets, i.e. $f_U(s)|_G V = f_V(s|_F V)$.

It is clear how to define the identity map $\mathrm{id}_F : F \to F$ of a presheaf and the composition $g \circ f$ of two homomorphims $f : F \to G$, $g : G \to H$ of presheaves.

There is also a natural notion of a sub-presheaf $F \subset G$. Besides $F(U) \subset G(U)$ for all U one has to demand, that the restrictions are compatible. This means:

The canonical inclusions $i_U : F(U) \to G(U)$ define a homomorphism $i : F \to G$ of presheaves.

When $f : F \to G$ is a homomorphism of presheaves, the images $f_U(F(U))$ define a sub-presheaf of G. We call it the *presheaf-image* and denote it by

$$f_{\mathrm{pre}}(F).$$

It is also clear that the kernels of the maps f_U define a sub-presheaf of F. We denote it by Kernel$(f : F \to G)$. When F is a sub-presheaf of G then one can can consider the factor groups $G(U)/H(U)$. Using II.1.1 it is clear how to define restriction maps to get a presheaf $G/_{\mathrm{pre}}F$. We call this presheaf the *factor-presheaf.*

Since we have defined Kernel and Image we can also introduce the notion of a *preasheaf-exact sequence.* A sequence $F \to G \to H$ is presheaf-exact if and only if $F(U) \to G(U) \to H(U)$ is exact for all U. What we have said about exact sequences of abelian groups carries literarily over to presheaf-exact sequences of presheaves of abelian groups.

2. Germs and Stalks

let F be a presheaf on a topological space X und let $a \in X$ be a point. We consider pairs (U, s), where U is an open neighpourhood of a and $s \in F(U)$ a section over U. Two pairs (U, s), (V, t) are called equivalent, if there exists an open neighborhood $a \in W \subset U \cap V$, such that $s|W = t|W$. This is an equivalence relation. The equivalence classes

$$[U, s]_a := \{\, (V, t); \quad (V, t) \sim (U, s) \,\}$$

are called *germs* of F in the point a. The set of all germs

$$F_a := \{\, [U, s]_a, \quad a \subset U \subset X,\ s \in F(U) \,\}$$

is the so-called *stalk* of F in a. The stalk carries a natural structure as abelian group. One defines

$$[U, s]_a + [V, t]_a := [U \cap V, s|U \cap V + t|U \cap V]_a.$$

We use frequently the simplified notation

$$s_a = [U, s]_a.$$

For every open neighborhood $a \in U \subset X$ there is an obvious homomorphism

$$F(U) \longrightarrow F_a, \quad s \longmapsto s_a.$$

A homomorphism of presheaves $f : F \to G$ induces natural mappings

$$f_a : F_a \longrightarrow G_a \qquad (a \in X).$$

The image of a germ $[U, s]_a$ is simply $[U, f_U(s)]_a$. It is easy to see that this is well-defined.

2.1 Remark. *Let $F \to G$ and $G \to H$ be homomorphism of presheaves and let $a \in X$ be a point. Assume that every neighborhood of a contains a small open neighborhood U such that $F(U) \to G(U) \to H(U)$ is exact. Then $F_a \to G_a \to H_a$ is exact.* Hpk

Corollary. *if $F \to G \to H$ is presheaf-exact then $f_a \to G_a \to H_a$ is exact for all a.*

If F is a preasheaf on X, one can consider for each open subset $U \subset X$

$$F^{(0)}(U) := \prod_{a \in U} F_a.$$

The elements are families $(s_a)_{a \in U}$ with $s_a \in F_a$. There is now coupling between the different s_a. Hence $F^{(0)}(U)$ usually is very giantly.

For open sets $V \subset U$, one has an obvious homomorphism $F^{(0)}(U) \to F^{(0)}(V)$. Hence we obtain a preasheaf $F^{(0)}$ together with a natural homomorphism

$$F \longrightarrow F^{(0)}.$$

3. Sheaves

3.1 Definition. *A presheaf F is called **sheaf**, if the following conditions are satisfied:* DG

(G1) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ and if $s, t \in F(U)$ are sections with the property $s|U_i = t|U_i$ fur alle i, then $s = t$.*

(G2) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ und if $s_i \in F(U_i)$ is a family of sections with the property*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j \quad \textit{fur alle } i, j,$$

then there exists a section $s \in F(U)$ with the property $s|U_i = s_i$ for all i.

(G3) *$F(\emptyset)$ is the zero group.*

Clearly the presheaf of continuous functions is a sheaf, since continuity is a local property. An example of a presheaf F, which usually is not a sheaf is the presheaf of constant functions with values in $\mathbb{Z}$ ($F(U) = \{f : U \to \mathbb{Z},\ f$ constant$\}$). But the set of *locally constant* functions with values in $\mathbb{Z}$ is a sheaf.

By a subsheaf of a sheaf F we understand a sub-presheaf $G \subset F$ which is already a sheaf. If F, G are presheaves then a homomorphism $f : F \to G$ of presheaves is called also a homomorphism of sheaves.

3.2 Remark. *Let $F \subset G$ be a sub-presheaf. We assume that G (but not necessarily F) is a sheaf. Then there is a smallest subsheaf $\tilde{F} \subset G$ which contains F. For an arbitrary point $a \in X$ the induced map $f_a : F_a \to \tilde{F}_a$ is an isomorphism.* Eug

It is clear, that $\tilde{F}(U)$ has to be defined as set of all $s \in G(U)$, such that:

There exists an open covering $U = \bigcup U_i$, such that $s|U_i$ is in the image of $F(U_i) \to G(U_i)$ for all i.

This is equivalent with:

The germ s_a is in the image of $F_a \to G_a$ for all $a \in U$.

3.3 Definition. *Let $F \to G$ be a homomorphism of sheaves. The sheaf-image $f_{sheaf}(F)$ is the smallest subsheaf of G, which contains the presheaf-image-$f_{pre}(F)$.* Bpg

We have to differ between two natural notions of surjectivity.

3.4 Definition. Sis

1) *A homomorphism of preasheaves $f : F \to G$ is called **presheaf-surjective** if $f_{pre}(F) = G$.*

2) *A homomorphism of sheaves $f : F \to G$ is called **sheaf-surjective** if $f_{sheaf}(F) = G$.*

Wenn F and G both are sheaves then sheaf-surjectivity and presheaf-surjectivity are different things. We give an example which will be basic:

Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$, hence $\mathcal{O}(U)$ is the set of all holomorphic functions on an open subset U. This a sheaf of abelian groups (under addition). Similarly we consider the sheaf $\mathcal{O}^*$ of holomorphic functions without zeros. This is also a sheaf of abelain groups (under multiplication). The map $f \to e^f$ defines a sheaf homomorphism

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*.$$

The map $\mathcal{O}(U) \to \mathcal{O}^*(U)$ is not always surjective. For example for $U = \mathbb{C}^*$ the function $1/z$ is not in the image. Hence exp is not presheaf-surjective. But it is know from complex calculus that $\exp : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is surjective if U is simply connected, for example for a disc U. Since a point admits arbitrarily small neighborhoods which are discs, it follows that exp is sheaf-surjective.

3.5 Remark. *A homomorphism of sheaves $f : F \to G$ is sheaf-surjective if and only if the maps $f_a : F_a \to G_a$ are surjective for all $a \in X$.* gH

Fortunately the notion "injective" doesn't contain this difficulty.

3.6 Remark. *Let $f : F \to G$ be a homomorphism of sheaves. The kernel in the sense of presheaves is already a sheaf.* KiG

Hence we don't have to distinguish between presheaf-injective and sheaf-injective and also not between preasheaf-kernel and sheaf-kernel.

3.7 Remark. *A homomorphism of sheaves $f : F \to G$ is injective if and only if the maps $f_a : F_a \to G_a$ ar injective for all $a \in X$.* hH

A homomorphism of presheaves $f : F \to G$ (sheaves) is called an isomorphism if all $F(U) \to G(U)$ are isomorphisms. Their inverses then define a homomorphism $f^{-1} : G \to F$.

3.8 Remark. *A homomorphism of sheafs $F \to G$ is an isomorphism if and only if $F_a \to G_a$ is an isomorphism for all a.* AGb

For presheaves this is false. As counter example on can take for F the presheaf of constant functions and for G the sheaf of locally constant functions.

It is natural to introduce the notion of sheaf-exactness as follows:

3.9 Definition. *A sequence $F \to G \to H$ of sheaf homomorphims is sheaf-exact at G , if the the kernel of $G \to H$ and the sheaf-image of $F \to G$ agree.* Dse

Generalizing 3.5 and 3.7 one can easily show:

3.10 Proposition. *A sequence $F \to G \to H$ is exact if and only if $F_a \to G_a \to H_a$ is exact for all a.* Pee

Our discussion so far has obviously one gap: Let $F \subset G$ be subsheaf of a sheaf G. We would like to have an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The sheaf H should be the factor sheaf of G by F. But up to now we only defined the factor-presheaf $G/_{\mathrm{pre}}F$ which usually is no sheaf. In the next section we will give the correct definition for a factor sheaf $G/_{\mathrm{sheaf}}F$.

4. The generated sheaf

For a presheaf F we introduced the monstrous presheaf

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

Obviously $F^{(0)}$ is a sheaf. Sometimes its is called the "Godement-sheaf" or the "associated flabby sheaf". There is a natural homomorphism

$$F \to F^{(0)}.$$

We can consider its presheaf-image and then the smallest subsheaf which contains it. We denote this sheaf by $\hat{F}$ and call it the "generated sheaf" by F. There is a natural homomorphism

$$F \to \hat{F}.$$

From the construction follows immediately

4.1 Remark. *Let F be a presheaf. The natural maps* ePi

$$F_a \xrightarrow{\sim} \hat{F}_a$$

are isomorphisms.

A homomorphism $F \to G$ of presheaves induces a homomorphism $F^{(0)} \to G^{(0)}$. Clearly $\hat{F}$ is mapped into $\hat{G}$.

4.2 Remark. *Let $f : F \to G$ be a homomorphism of presheaves. There is a natural homomorphism $\hat{F} \to \hat{G}$, such that the diagram* UEg

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

commutes.

When F is already a sheaf then $F \to F^{(0)}$ is injective. Then the map of F into the presheaf image is an isomorphism. This implies that the presheaf image is already a sheaf.

4.3 Remark. *Let F be a sheaf. Then $F \to \hat{F}$ is an isomorphism.* FiF

If F is a sub-presheaf of a sheaf G, then the induced map $\hat{F} \to \hat{G} \cong G$ is an isomorphism $\hat{F} \to \tilde{F}$ between $\hat{F}$ and the smallest subsheaf $\tilde{F}$ of G, wich contains F.

We identify $\tilde{F}$ and $\hat{F}$.

Factor sheaves and exact sequences of sheaves

Let $F \to G$ be a homomorphism of presheaves. We introcuced already the factor preshaf $G/_{\text{pre}}F$, which asociates to an open U the factor group $G(U)/F(U)$. Even if both F and G are sheaves this will usually not a shesf. Hence we define the factor sheaf as the sheaf generated by the factor-presheaf.

$$G/_{\text{sheaf}}F := \widehat{G/_{\text{pre}}}F.$$

This called the factor-sheaf. Since we are interested mainly in sheaves, we will write usually for a homomorphism for sheaves $f : F \to G$:

$$\begin{aligned} G/F &:= G/_{\text{sheaf}}F && \text{(factor sheaf)} \\ f(F) &:= f_{\text{sheaf}}(F) && \text{(sheaf image)} \end{aligned}$$

Notice that there is no need to differ between sheaf- and presheaf-kernel. When we talk about an exact sequence of sheaves

$$F \longrightarrow G \longrightarrow H$$

we usually mean "sheaf exactness". All what we have said about exactness properties of sequences of abelian groups is literally true for sequences of sheaves. For example: A sequence of sheaves $0 \to F \to G$ (0 denotes the zero sheaf) is exact if and only of $F \to G$ is injective. A sequence of sheaves $F \to G \to 0$ is exact if and only if $F \to G$ is surjective (in the sense of sheaves of course). A sequence of sheaves $0 \to F \to G \to H \to 0$ is exact if and only if there is an ismomorphism $H \cong G/F$ which identifies this sequence with

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0.$$

4.4 Remark. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves. Then for open U the sequence* ESf

$$0 \to F(U) \to G(U) \to H(U)$$

is exact.

Corollary. *The sequence*

$$0 \to F(X) \to G(X) \to H(X)$$

is exact.

Usually $G(X) \longrightarrow H(X)$ is not surjective as the example

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O} \stackrel{f \mapsto e^{2\pi \mathrm{i} f}}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

shows. Cohomology theory will measure the absence the right exactness. The above sequence will be part of a long exact sequence

$$0 \to F(X) \to G(X) \to H(X) \longrightarrow H^1(X,F) \longrightarrow \cdots$$

Chapter IV. Cohomology of sheaves

1. The canonical flabby resolution

A sheaf F is called *flabby,* if $F(X) \to F(U)$ is surjective of all U. Then $F(U) \to F(V)$ is surjective for all $V \subset U$. An example for a flabby sheaf is the Godement sheaf $F^{(0)}$. Recall that we have the exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}.$$

We want to extend this sequence. For this we consider the sheaf $F^{(0)}/F$ and embed it into its Godement sheaf,

$$F^{(1)} := \left(F^{(0)}/F\right)^{(0)}.$$

In this way we get a long exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots$$

If $F^{(n)}$ has been already constructed then we define

$$F^{(n+1)} := \left(F^{(n)}/F^{(n-1)}\right)^{(0)}.$$

The sheaves $F^{(n)}$ are all flabby. We call this sequence the *canonical flabby resolution* or the *Godement resolution.* Sometimes it is useful to write the resolution in the form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{(0)} & \longrightarrow & F^{(1)} & \longrightarrow & F^{(2)} & \longrightarrow & F^{(3)} & \longrightarrow & \cdots
\end{array}$$

Both lines are complexes. The vertical arrows can be considered as a complex homomorphism. The induced homomorphism of the cohomology groups are isomorphisms. Notice that only the 0-cohomology group of both complexes is different from 0. This zero cohomology group is naturally isomorphic F.

Now we apply the global section functor Γ to the resolution. This is

$$\Gamma F := F(X).$$

We obtain a long sequence

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots$$

The essential point is that this sequence is no longer exact. we only can say that it is a complex. We prefer to write in the form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \Gamma F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \Gamma F^{(0)} & \longrightarrow & \Gamma F^{(1)} & \longrightarrow & \Gamma F^{(2)} & \longrightarrow & \Gamma F^{(3)} & \longrightarrow & \cdots
\end{array}$$

The second line is

$$\begin{array}{ccccccccc}
\cdots \longrightarrow 0 \longrightarrow & \Gamma F^{(0)} & \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots \\
 & \uparrow & \\
 & \text{zero position} &
\end{array}$$

Now we define the cohomology groups $H^\bullet(X, F)$ to be the cohomology groups of this complex:

$$H^n(X, F) := \frac{\mathrm{Kern}(\Gamma F^{(n)} \longrightarrow \Gamma F^{(n+1)})}{\mathrm{Kern}(\Gamma F^{(n-1)} \longrightarrow \Gamma F^{(n)})}$$

(We define $\Gamma F^{(n)} = 0$ for $n < 0$.) Clearly

$$H^n(X, F) = 0 \qquad \text{for} \quad n < 0.$$

Next we treat the special case $n = 0$,

$$H^0(X, F) = \mathrm{Kernel}(\Gamma F^{(0)} \longrightarrow \Gamma F^{(1)}).$$

Since the kernel can be taken in the presheaf sense, we can write

$$H^0(X, F) = \Gamma\mathrm{Kernel}(F^{(0)} \longrightarrow F^{(1)}).$$

Recall that $F^{(1)}$ is a sheaf, which contains $F^{(0)}/F$ as subsheaf. We obtain

$$H^0(X, F) = \Gamma\mathrm{Kernel}(F^{(0)} \longrightarrow F(0)/F)$$

This is the image of F in $F^{(0)}$ an hence a sheaf which is canonically isomorphic to F.

1.1 Remark. *There is a natural isomorphism* sNI

$$H^0(X,F) \cong \Gamma F = F(X).$$

If $F \to G$ is a homomorphism of sheaves, then the homomorphism $F_a \to G_a$ induce a homomorphism $F^{(0)} \to G^{(0)}$. If $F \to G \to H$ is an exact sequence. Then $F^{(0)} \to G^{(0)} \to H^{(0)}$ is also exact (already as sequence of presheaves). More generally

1.2 Lemma. *Let $0 \to F \to G \to H \longrightarrow 0$ be an exact sequence of sheaves. Then the induced sequence $0 \to F^{(n)} \to G^{(n)} \to H^{(n)} \to 0$ is exact for every n.* Gae

The proof is by induction. One needs the following lemma about abelian groups:

Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram such that the three columns and the first to lines are exact. Then the third line is also exact.

This follows from II.2.2. □

Before we continue we need a basic lemma:

1.3 Lemma. *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves. Assume that F is flabby. Then* Wre

$$0 \to \Gamma F \to \Gamma G \to \Gamma H \to 0$$

is exact.

Proof. Let $h \in H(X)$. We have to show that h is the image of an $g \in G(X)$. For the proof one considers the set of all pairs (U, g), where U is an open subset and $g \in G(U)$ and such that g maps to $h|U$. This set is ordered by

$$(U, g) \geq (U', g') \Longleftrightarrow U' \subset U \text{ and } g|U' = g'.$$

From the sheaf axioms follows that every inductive subset has an upper bound. By Zorns's lemma there exists a maximal (U, g). We have to show $U = X$. If this is not the case, we can find a pair (U', g') in the above set such that U' is not contained in U. The difference $g - g'$ defines a section in $F(U \cap U')$. Since F is flabby, this extends to a global section. This allows us to modify g' such that it glues with g to a section on $U \cup U'$. □

An immediate corollary of 1.3 states:

1.4 Lemma. *Let $0 \to F \to G \to H \to 0$ an exact sequence of sheaves. If F and G are flabby then H is flabby too.* Wrf

Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheafs. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n-1)} & \longrightarrow & G^{(n-1)} & \longrightarrow & H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n)} & \longrightarrow & G^{(n)} & \longrightarrow & H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n+1)} & \longrightarrow & G^{(n+1)} & \longrightarrow & H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

From 1.2 we know that all lines of this diagram are exact From 1.3 follows that they remain exact after applying Γ. Hence the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n-1)} & \longrightarrow & \Gamma G^{(n-1)} & \longrightarrow & \Gamma H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n)} & \longrightarrow & \Gamma G^{(n)} & \longrightarrow & \Gamma H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n+1)} & \longrightarrow & \Gamma G^{(n+1)} & \longrightarrow & \Gamma H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

can be considered as a short exact sequence of complexes. We can apply II.2.1 to obtain the long exact cohomology sequence:

1.5 Theorem. *Every short exact sequence* $0 \to F \to G \to H \to 0$ *induces a natural long exact cohomology sequence* Less

$$0 \to \Gamma F \longrightarrow \Gamma G \longrightarrow \Gamma H \xrightarrow{\delta} H^1(X,F) \longrightarrow H^1(X,G) \longrightarrow H^1(X,H) \xrightarrow{\delta} H^2(X,F) \longrightarrow \cdots$$

The next Lemma shows that the cohomology of flabby sheaves is trivial.

1.6 Lemma. *Let* Wsa

$$0 \to F \longrightarrow F_0 \to F_1 \longrightarrow \cdots$$

be an exact sequence of flabby sheaves (finite or infinite). Then

$$0 \to \Gamma F \longrightarrow \Gamma F_0 \to \Gamma F_1 \longrightarrow \cdots$$

is exact.

Corollary. *For flabby* F *one has:*

$$H^i(X,F) = 0 \quad \textit{for} \quad i > 0.$$

Proof. We use the so-called splitting principle. The long exact sequence can be splitted into short exact sequences

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_0/F \longrightarrow 0, \quad 0 \longrightarrow F_0/F \longrightarrow F_1 \longrightarrow F_1/F_0 \longrightarrow 0, \ldots .$$

From 1.4 we get that the $F_0/F, F_1/F_0, \ldots$ are flabby. The claim now follows from 1.3. □

A sheaf F is called *acyclic* if $H^n(X,F) =$ for $n > 0$. Hence flabby sheaves are acyclic. By an *acyclic* resolution of a sheaf we understand an exact sequence

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

with acyclic F_i.

1.7 Proposition. *Let* $0 \to F \to F_0 \to F_1 \to \cdots$ *be an acyclic resolution of* F*. Then there is a natural isomorphism between the* n*-the cohomology group* $H^n(X,F)$ *and the* n*-th cohomology group of the complex* BmA

$$\cdots \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ \textit{zero position}}}{\Gamma F_0} \longrightarrow \Gamma F_1 \longrightarrow \Gamma F_2 \longrightarrow \cdots$$

Proof. Taking the canonical flabby resolutions of F and of all F_n on gets a diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(0)} & \longrightarrow & F_0^{(0)} & \longrightarrow & F_1^{(0)} & \longrightarrow & F_2^{(0)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(1)} & \longrightarrow & F_0^{(1)} & \longrightarrow & F_1^{(1)} & \longrightarrow & F_2^{(1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(2)} & \longrightarrow & F_0^{(2)} & \longrightarrow & F_1^{(2)} & \longrightarrow & F_2^{(2)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

All lines and columns are exact. We apply Γ to this complex. Then all lines and columns besides the first ones remain exact. We can apply II.2.2. $\square$

One may ask what "natural" means in 1.7 means. It means that certain diagrams in which this ismomorphism appears are commutative. Since it is the best to check this when it is used we give just one example: Consider the above commutative diagram in the following new meaning: All occurring sheaves besides F are acyclic. Then II.2.2 gives an isomorphism between the n-th cohomology groups of the complexes $0 \to \Gamma F_0 \to \Gamma F_1 \to \cdots$ and $0 \to \Gamma F^{(0)} \to \Gamma F^{(1)} \to \cdots$. Both are isomorphic to $H^n(X, F)$. This gives a commutative triangle.

2. Sheaves of rings and modules

A sheaf of A-modules is a sheaf F of abelian groups such that every $F(U)$ carries a structure as A-module and such the the restriction maps $F(U) \to F(V)$ for $V \subset U$ are A-linear. A homomorphism $F \to G$ is called A-linear if all $F(U) \to G(U)$ are so. Then kernel and image carry natural structures of sheafs of A-modules. Also the stalks carry such a structure naturally. Hence the whole canonical flabby resolution is a sequence of sheafs of A-modules. This implies that the cohomology groups also are A-modules.

There is a refinement of this construction: By a sheaf of rings $\mathcal{O}$ we understand a sheaf of abelian groups such that every $\mathcal{O}(U)$ is not only an abelian group but a ring and such that all restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$ are ring homomorphisms. Then the stalks $\mathcal{O}_a$ carry a natural ring structure such that the homomorphisms $\mathcal{O}(U) \longrightarrow \mathcal{O}_a$ (U is an open neighborhood of a) are ring homomorphisms.

By an $\mathcal{O}$-module we understand a sheaf $\mathcal{M}$ of abelian groups such every $F(U)$ carries a structure as $\mathcal{O}(U)$-module and such that the restriction maps are compatible with the module structure. To make this precise we give a short comment. Let M be an A-module and N be a module over a different ring B. Asssume that a homomorphism $r : A \to B$ is given. A homomorphism $f : M \to N$ of abelian groups is called compatible with the module structures if the formula

$$f(am) = r(a)f(m) \qquad (a \in A,\ m \in M)$$

holds. An elegant way to express this is as follows. We can consider N also as an module over A by means of the definition $an := r(a)n$. Sometimes this A-module is written as $N_{[r]}$. Then the compatibility of the map f simply means that it is an A-linear map

$$f : M \longrightarrow N_{[r]}.$$

Usually we will omit the subscript $[r]$ and simply say that $f : M \to N$ is A-linear.

If $\mathcal{M}$ is an $\mathcal{O}$-module then the stalk $\mathcal{M}_a$ is naturally an $\mathcal{O}_a$-module. An $\mathcal{O}$-linear map $f : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{O}$-modules is a homomorphism of sheaves of abelian groups such the maps $\mathcal{M}(U) \to \mathcal{N}(U)$ are $\mathcal{O}(U)$ linear. Then the Kernel and image also carry natural structures of $\mathcal{O}$-modules. Clearly the canonical flabby resolution of an $\mathcal{O}$-module is naturally a sequence of $\mathcal{O}$-modules.

Since for every open subset $U \subset X$ we have a ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$ all $\mathcal{M}(U)$ can be considered as $\mathcal{O}(X)$-modules. Hence a $\mathcal{O}$-module can be considered as sheaf of $\mathcal{O}(X)$-modules. Especially $H^n(X, \mathcal{M})$ carries a natural structure as $\mathcal{O}(X)$-module.

We consider a very special case. We take for $\mathcal{O}$ the sheaf $\mathcal{C}$ of continuous functions. There are two possibilities: $\mathcal{C}_{\mathbb{R}}$ is the sheaf of continuous real-valued and $\mathcal{C}_{\mathbb{C}}$ the sheaf of continuous complex-valued functions. If we write $\mathcal{C}$ we mean one of both. The sheaf $\mathcal{C}$ or more generally a module over this sheaf have over paracompact spaces a property which can be considered as a weakend form of flabbyness.

2.1 Remark. *Let X be paracompact space and $\mathcal{M}$ a $\mathcal{C}$-module on X. Assume* PW
that U is an open subset and $V \subset\subset U$ an open subset which is relatively compact in U. Assume that $s \in \mathcal{M}(U)$ is a section over U. Then there is a global section $S \in \mathcal{M}(X)$ such that $S|V = s|V$.

Proof. We choose a continuous real valued function φ on X, which is one on V and whose support is compact and contained in U. Then we consider the open covering $X = U \cup U'$, where U' denotes the complement of the support of φ. On U we consider the section φs and on U' the zero section. Since both are zero on $U \cap U'$ they glue to a section S on X. $\square$

2.2 Lemma. *Let X be a paracompact space and $\mathcal{M} \to \mathcal{N}$ a surjective $\mathcal{C}$-linear map of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X)$ is surjective.* cCs

Proof. Let $s \in \mathcal{N}(X)$. There exists an open covering $(U_i)_{i\in I}$ of X such that $s|U_i$ is the image of an section $t_i \in \mathcal{M}(U_i)$. We can assume that the covering is locally finite. We take relatively compact open subsets $V_i \subset U_i$ such that (V_i) is still a covering. Then we choose a partition of unity (φ_i) with respect to (V_i). By 2.2 there exists global sections $T_i \in \mathcal{M}(X)$ with $T_i|V_i = t_i|V_i$. We now consider

$$T := \sum_{i\in I} \varphi_i T_i.$$

Since I can be infinite we have to explain what this means. Let $a \in X$ a point. There exists an open neighborhood $U(a)$ such $V_i \cap U(a) \neq \emptyset$ only for a finite subset $J \subset I$. We can define the section

$$T(a) := \sum_{i\in J} \varphi T_i|U(a).$$

The sets $U(a)$ cover X and the sections $T(a)$ glue to a section T. Clearly T maps to s. $\square$

2.3 Lemma. *Let X be a paracompact space and $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ an exact sequence of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X) \to \mathcal{P}(X)$ is exact too.* Pbc

Proof. The exactness of the sequence implies the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M} \to \mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \text{Kernel}(\mathcal{N} \to \mathcal{P}) \longrightarrow 0.$$

From 2.2 we get

$$0 \longrightarrow \text{Image}(\mathcal{M} \to \mathcal{N})(X) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N} \to \mathcal{P})(X) \longrightarrow 0.$$

Applying 2.2 to $\mathcal{M} \to \text{Image}(\mathcal{M} \to \mathcal{N})$ we obtain

$$\text{Image}(\mathcal{M} \to \mathcal{N})(X) = \text{Image}(\mathcal{M}(X) \to \mathcal{N}(X)).$$

Since also

$$\text{Kernel}(\mathcal{N} \to \mathcal{P})(X) = \text{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X))$$

we get the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M}(X) \to \mathcal{N}(X)) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X)) \longrightarrow 0.$$

This proves 2.3. $\square$

Let $\mathcal{M}$ b an $\mathcal{C}$-module over a paracompact space. Then the canonical flabby resolution is also a sequence of $\mathcal{C}$-modules. From 2.3 follows that the resolution remains exact after the application of Γ. We obtain.

2.4 Proposition. *Let X be paracompact. Every $\mathcal{C}$-module is acyclic, i.e. $H^n(X,\mathcal{M}) = 0$ for $n > 0$.* PcC

The essential tool of the proofs has been the existence of a partition of unity. Partitions of unity exist also in the differentiable world. Hence there is the following variant of 2.3.

2.5 Proposition. *Let X be a paracompact differentiable manifold, then every $\mathcal{C}^\infty$-modul is acyclic.* DPa

3. Čech Cohomology

Here we will consider only the first Čech cohomology group of a sheaf. We have to work with open coverings $\mathfrak{U} = (U_i)_{i\in I}$ of the given topological space X. Let F be sheaf on X. A one-cocycle of F with respect to the covering $\mathfrak{U}$ is family of sections

$$s_{ij} \in F(U_i \cap U_j), \quad (i,j) \in I \times I,$$

with the following property: For each triple i,j,k of indices one has

$$s_{ik} = s_{ij} + s_{jk} \quad \text{on} \quad U_i \cap U_j \cap U_k.$$

In more precise writing this means

$$s_{ik}|(U_i \cap U_j \cap U_k) = s_{ij}|(U_i \cap U_j \cap U_k) + s_{jk}|(U_i \cap U_j \cap U_k).$$

We denote by $C^1(\mathfrak{U}, F)$ the group of all one-cocycles. Assume that a family of sections $s_i \in F(U_i)$ is given. Then

$$s_{ij} = s_i|(U_i \cap U_j) - s_j|(U_i \cap U_j)$$

obviously is a cocycle. We denote it by

$$\delta(s_i)_{i\in I}.$$

A cocycle of this form is called a coboundary. The set of all coboundaries is a subgroup

$$B^1(\mathfrak{U}, F) \subset C^1(\mathfrak{U}, F).$$

The (first) Čech cohomology of F with respect to the covering $\mathfrak{U}$ is defined as

$$\check{\mathrm{H}}^1(\mathfrak{U}, F) := C^1(\mathfrak{U}, F)/B^1(\mathfrak{U}, F).$$

A homomorphism of sheaves $F \to G$ induces a homomorphism

$$\check{\mathrm{F}}^1(\mathfrak{U}, F) \longrightarrow \check{\mathrm{G}}^1(\mathfrak{U}, F).$$

Let $f : G \to H$ be a surjective homomorphism of sheaves and $\mathfrak{U} = (U_i)$ an open covering of X. We denote by $H_{\mathfrak{U}}(X)$ the set of all global sections of H with the following property:

For every index i there is a section $t_i \in G(U_i)$ with $f(t_i) = s|U_i$. By definition of (sheaf-)surjectivity for every global section $s \in H(X)$ there exists an open covering $\mathfrak{U}$ with $s \in H_{\mathfrak{U}}(X)$. It follows

$$H(X) = \bigcup_{\mathfrak{U}} H_{\mathfrak{U}}(X).$$

Let $0 \to F \to G \to H \to 0$ be an exact sequence and $\mathfrak{U}$ an open covering. There exists a natural homomorphism

$$\delta : H_{\mathfrak{U}}(X) \longrightarrow \check{H}^1(U, F),$$

which is constructed as follows: Let be $s \in H_{\mathfrak{U}}(X)$. We choose elements $t_i \in G(U_i)$ which are mapped to $s|U_i$. The differences $t_i - t_j$ come from sections $t_{ij} \in F(U_i \cap U_j)$. They define a 1-cocycle $\delta(s)$. It is easy to check that this corresponding element of $\check{H}^1(\mathfrak{U}, F)$ doesn't depend on the choice of the t_i.

3.1 Lemma. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves and $\mathfrak{U}$ an open covering. The sequence* Cei

$$0 \longrightarrow F(X) \longrightarrow G(X) \longrightarrow H_{\mathcal{U}}(X) \xrightarrow{\delta} \check{H}^1(\mathcal{U}, F) \longrightarrow \check{H}^1(\mathcal{U}, G) \longrightarrow \check{H}^1(\mathcal{U}, H)$$

is exact.

Remark. *This sequence doesn't extend naturally to a long sequence.*

This Lemma indicates that Čech cohomology must be related to usual cohomology. Another result in this direction is:

3.2 Remark. *Let F be a flabby sheaf. Then for every open covering* Cwv

$$\check{H}^1(\mathfrak{U}, F) = 0.$$

Proof. We start with a little remark. Assume that the whole space $X = U_{i_0}$ is a member of the covering. Then the Čech cohomology vanishes (for every sheaf): If (s_{ij}) is a cocycle one defines $s_i = s_{i,i_0}$. Then $\delta((s_i)) = (s_{ij})$. For the proof of 3.2 we now consider the sequence

$$\begin{array}{ccccccc}
0 \longrightarrow F(X) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ij} F(U_i\cap U_j) & \longrightarrow & \prod_{ijk} F(U_i\cap U_j\cap U_k) \\
s & \longmapsto & (s|U_i) & & & & \\
 & & (s_i) & \longmapsto & (s_i - s_j) & & \\
 & & & & (s_{ij}) & \longmapsto & (s_{ij}+s_{jk}-s_{ik})
\end{array}$$

We will proof that this sequence is exact. (Then 3.2 follows.) The idea is to sheafify this sequence: For an open subset $U \subset X$ one considers $F|U$ and also the restricted covering $U \cap U_i$. Repeating the above construction for U instead of X on obtains a sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}.$$

Since F is flabby, also $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are flabby. The remark at the beginning of the proof shows that $0 \longrightarrow F(U) \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{C}(U)$ is exact, when U is contained in some U_i. Hence the sequence of sheaves is exact. From 1.6 follows that the exactness is also true for $U = X$. □

Let now F be an arbitrary sheaf, $F^{(0)}$ the associated flabby sheaf. We get an exact sequence $0 \to F \to F^{(0)} \to H \to 0$. let $\mathfrak{U}$ be an open covering. We know that $\check{H}^1(\mathcal{U}, F^{(0)})$ vanishes, 3.2. From 3.1 we obtain an isomorphy

$$\check{H}^1(\mathcal{U}, F) \cong H_{\mathfrak{U}}(X)/G(X).$$

From the long exact cohomology sequence we get for the usual cohomology

$$H^1(X, F) \cong H(X)/G(X).$$

This gives an *injective* homomorphism

$$\check{H}^1(\mathcal{U}, F) \longrightarrow H^1(X, F).$$

This gives us:

3.3 Proposition. *Let F be a sheaf. Then* Cgr

$$H^1(X, F) = \bigcup_{\mathcal{U}} \check{H}^1(\mathcal{U}, F).$$

The following commutative daigram that the Čech combining δ from 3.1 and that of general sheaf theory 1.5 coincide:

3.4 Remark. *For a short exact sequence* $0 \to F \to G \to H \to 0$ *the diagram* CdSd

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H_{\mathfrak{U}}(X) & \xrightarrow{\delta} & \check{H}^1(\mathfrak{U}, F) \\ & & \| & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) & \xrightarrow{\delta} & H^1(X, F) \end{array}$$

is commutative.

The proof is left to the reader. □

Let $\mathfrak{V} = (V_j)_{j \in J}$ be a refinement of $\mathfrak{U} = (U_i)_{i \in I}$ and $\varphi : J \longrightarrow I$ a refinement map ($V_\varphi \subset U_i$). Using this refinement map one obtains a natural map

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{V}, F).$$

This shows:

3.5 Remark. *Let* $\mathfrak{V}$ *be an refinement of* $\mathfrak{U}$ *and* $\varphi : J \to I$ *a refinement map.* Vua
The diagram

$$\begin{array}{ccc} \check{H}^1(\mathfrak{U}, F) & \longrightarrow & \check{H}^1(\mathfrak{V}, F) \\ & \searrow \quad \swarrow & \\ & H^1(X, F) & \end{array}$$

commutes. Especially it doesn't depend on the choice of the refinement map.

Usually it is of course very difficult to control all open coverings of a topological space. But sometimes a single covering is sufficient:

3.6 Theorem of Leray. *Let* F *be a sheaf on* X *and* $\mathfrak{U} = (U_i)$ *an open* ToL
covering of X*. Assume that* $H^1(U_i, F|U_i) = 0$ *for all* i*. Then*

$$H^1(X, F) = \check{H}^1(\mathfrak{U}, F).$$

Proof. Since two coverings admit a joint refinement it is sufficient to prove that $\check{H}^1(\mathfrak{U}, F) \to \check{H}^1(\mathfrak{V}, F)$ is an isomorphism for each refinement $\mathfrak{V}$ of $\mathfrak{U}$. Since the map is injective, it remains to prove surjectivity. We choose a refinement map $\varphi : J \to I$. We denote the indices form I by $i, j, \ldots$ and those of J by $\alpha, \beta, \ldots$. Let be $(s_{\alpha,\beta}) \in C^1(\mathfrak{V}, F)$. We consider the covering $U_i \cap \mathfrak{V} := (U_i \cap V_\alpha)_\alpha$ of U_i. From the assumption $\check{H}^1(U_i \cap \mathfrak{V}, F|U_i) = 0$ we get the existence of $t_{i\alpha} \in F(U_i \cap V_\alpha)$ such that

$$s_{\alpha\beta} = t_{i\alpha} - t_{i\beta} \quad \text{on} \quad U_i \cap V_\alpha \cap V_\beta.$$

From this equation follows that

$$t_{j\alpha} - t_{j\alpha} = t_{j\beta} - t_{j\beta} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha \cap V_\beta.$$

Hence these differences glue to section $T_{ij} \in F(U_i \cap U_j)$,

$$T_{ij} = t_{i\alpha} - t_{j\alpha} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha.$$

Clearly (T_{ij}) is a cocycle in $C^1(\mathfrak{U}, F)$. We consider its image $(T_{(\varphi\alpha,\varphi\beta)})$ in $C^1(\mathfrak{V}, F)$. It is easy to check that this cocycle and the one we started with $(s_{\alpha\beta})$ define the same cohomology class: The differ by the coboundary $(h_\beta - h_\alpha)$ with $h_\alpha = t_{\varphi\alpha,\alpha} \in F(V_\alpha)$. □

4. Some vanishing results

Let X be a topological space and A an abelian group. We denote by A_X the sheaf of locally constant functions with values in A. This sheaf can be identified with the sheaf which is generated by the presheaf of constant functions. We will write

$$H^n(X, A) := H^n(X, A_X).$$

4.1 Proposition. *Let U be an open and convex subset of $\mathbb{R}^n$. Then for every* Kkv
abelian group A

$$H^1(U, A) = 0.$$

Actually this is true for all H^n, $n > 0$. The best way to prove this to use the comparison theorem with singular cohomology as defined in algebraic topology. We restrict to H^1.

Proof of 4.1. Every convex open subset of $\mathbb{R}^n$ is topologically equivalent to $\mathbb{R}^n$. Hence it is sufficient to restrict to $U = \mathbb{R}^n$. Just for simplicity we assume $n = 1$. (The general case should then be clear.) We use Čech cohomology and show that every open covering admits a refinement $\mathfrak{U}$ such that $H^1(\mathfrak{U}, A_X) = 0$. To show this we take a refinement of a very simple nature. It is easy to show that there exists a refinement of the following form. The index set is $\mathbb{Z}$. There exists a sequence of real numbers (a_n) with the following properties:

a) $a_n \le a_{n+1}$
b) $a_n \to +\infty$ for $n \to \infty$ and $a_n \to -\infty$ for $n \to -\infty$
c) $U_n = (a_n, a_{n+2})$.

Assume that $s_{n,m}$ is a cocycle with respect to this covering. Notice that U_n has non empty intersection only with U_{n-1} and U_{n+1}. Hence only $s_{n-1,n}$ is of relevance. This a locally constant function on $U_{n-1} \cap U_n = (a_n, a_{n+1})$. Since this is connected, the function $s_{n-1,n}$ is constant. We want to show that it is coboundary, i.e. we want to construct constant functions s_n on U_n such that $s_{n-1,n} = s_n - s_{n-1}$ on (a_n, a_{n+1}). This is easy. One starts with $s_0 = 0$ and then constructs inductivly $s_1, s_2, \dots$ and in the same way for negative n. □

Consider on the real line $\mathbb{R}$ the sheaf of real valued differentiable functions $\mathcal{C}^\infty$. Taking derivatives one gets a sheaf homomorphism $\mathcal{C}^\infty \to \mathcal{C}^\infty$, $f \mapsto f'$. The kernel is the sheaf of all locally constant functions, which we denote simply by $\mathbb{R}$. Hence we get an sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty \longrightarrow 0.$$

This sequence is exact since every differentiable function has an integral. Hence this sequence can be considered as acyclic resolution of $\mathbb{R}$. We obtain $H^n(\mathbb{R}, \mathbb{R}) = 0$ for all $n > 0$. For $n = 1$ this follows already from 4.1. There is a generalization to higher dimensions. For example a standard result of vector analysis states in the case $n = 2$.

4.2 Lemma. *Let $E \subset \mathbb{R}^n$ be an open and convex subset, $f, g \in \mathcal{C}^\infty$ a pair of differentiable functions with the property* LP

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Then there is a differentiable function h with the property

$$f = \frac{\partial h}{\partial x}, \quad g = \frac{\partial h}{\partial y}.$$

In the sequence of exact sequences this means:
The sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{C} \longrightarrow & \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) & \longrightarrow 0 \\ & f & \longmapsto & \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) & & & \\ & & & (f, g) & \longmapsto & \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} & \end{array}$$

is exact. When E is not convex, this sequence needs not to be exact. But since every point in $\mathbb{R}^2$ has an open convex neighborhood, the sequence of sheaves

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \times \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \longrightarrow 0$$

is exact for every open subset $X \subset X$. This is an acyclic resolution and we obtain:

4.3 Satz. *For convex open $E \subset \mathbb{R}^2$ we have* Klc

$$H^i(E, \mathbb{R}) = 0 \quad \textit{for} \quad i > 0.$$

Who is familiar with alternating differential forms will see that the sequence is a special case of the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow A_X^0 \longrightarrow A_X^1 \longrightarrow \cdots \longrightarrow A_X^n \longrightarrow 0$$

Here X is a differentiable manifold of dimension n and A_X^i denotes the sheaf of alternating differential forms of degree i. The Lemma of Poincarè states that one gets an exact sequence, if one takes global sections on an open subset which is diffeomorphic to an open convex domain in $\mathbb{R}^n$. Hence this sequence is an acyclic resolution of $\mathbb{R}_X$ for arbitrary X We obtain

4.4 Theorem of de Rham. *For a differentiable manifold X on has* TdR

$$\dim H^i(X, \mathbb{C}_X) \cong \frac{\operatorname{Kern}(A^i(X) \longrightarrow A^{i+1}(X))}{\operatorname{Image}((A^{i-1}(X) \longrightarrow A^i(X))}.$$

Applying the Lemma of Poincarè again we obtain:

4.5 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has* Klz

$$H^i(E, \mathbb{R}_X) = 0 \quad \textit{fur} \quad i > 0.$$

Differential forms can also be considered complex valued. The Lemma of Poincarè remains true by trivial reasons. Hence we see also:

4.6 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has* Klze

$$H^i(E, \mathbb{C}_X) = 0 \quad \textit{fur} \quad i > 0.$$

As an application we prove

4.7 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has* Klzz

$$H^2(E, \mathbb{Z}) = 0.$$

Proof. We consider the homomorphism

$$\mathbb{C} \longrightarrow \mathbb{C}^\bullet, \qquad z \longmapsto e^{2\pi i z}.$$

The kernel is $\mathbb{Z}$. This can be considered as a exact sequence of sheaves for example on an open convex $E \subset \mathbb{R}^n$. A small part of the long exact cohomology sequence is

$$H^1(E, \mathbb{C}^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(E, \mathbb{C}).$$

Since the first and the third member of this sequence vanish (4.1 and 4.3) we get the proof of 4.5. □

Next we treat an example of complex analysis. For this we need

4.8 Lemma of Dolbeault. *Let $E \subset \mathbb{C}$ be an open disc. For every function* LvD
$f \in \mathcal{C}^\infty(E)$ there exists a $g \in \mathcal{C}^\infty(E)$ with

$$f = \frac{\partial g}{\partial \bar{z}} := \frac{\partial g}{\partial x} + \mathrm{i}\frac{\partial g}{\partial x}.$$

(s. Forster, Riemann surfaces, 13.2). The lemma of Dolbeault is related to an exact sequence of sheaves. Recall that a (real-) differentiable function is analytic (=holomorphic) if and only if $\partial f/\bar{\partial} z = 0$. We denote by $\mathcal{O}$ the sheaf of holomorphic functions. The Lemma of Dolbault shows that the sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}_X^\infty \stackrel{\bar{\partial}}{\longrightarrow} \mathcal{C}_X^\infty \longrightarrow 0$$

is exact for an open subset $X \subset \mathbb{C}$. This is an acyclic resoution of $\mathcal{O}_X$. Applying the Lemma of Dolbault once more, we get:

4.9 Satz. *Let $E \subset \mathbb{C}$ be an open disc. Then* KOe

$$H^i(E, \mathcal{O}_E) = 0 \quad \textit{for} \quad i > 0.$$

We denote by $\bar{\mathbb{C}}$ the Riemann sphere.

4.10 Theorem. *One has* KOk

$$H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}}) = 0.$$

For the proof we consider a covering of $\bar{\mathbb{C}}$ by two discs E_1, E_2 such that the intersection is a circular ring $1 < |z| < 2$. By Leray's theorem and the vanishing result 4.10 it is sufficient to show that the Čech cohomology with respect to this covering vanishes. A Čhech cocycle simply is given by a holomorphic function on the circular ring. We have to show that it can be written as difference $f_1 - f_2$ where f_i is holomorphic on the disc E_i. This is possible by the theory of the *Laurent decomposition.* □

Chapter V. Basic facts about Riemann surfaces

1. Geometric spaces

1.1 Definition. *A geometric structure $\mathcal{O}$ on a topological space is a collection of subrings $\mathcal{O}(U) \subset \mathcal{C}(U)$, where U runs through all open subsets, such that the following conditions are satisfied:* DgR

1. *The constant functions are in $\mathcal{O}(U)$.*
2. *If $V \subset U$ are open sets then*

$$f \in \mathcal{O}(U) \Longrightarrow f|V \in \mathcal{O}(V).$$

3. *Let $(U_i)_{i\in I}$ be a system of open subsets and $f_i \in \mathcal{O}(U_i)$ such that*

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j \quad \textit{for all} \quad (i,j),$$

then there exist a $f \in \mathcal{O}(U)$ where $U = \bigcup_{i\in I} U_i$ with the property

$$f|U_i = f_i \quad \textit{for all} \quad i.$$

We see that $\mathcal{O}$ is sheaf of rings. We call the functions of $\mathcal{O}(U)$ the distinguished functions. Conditions two and three mean that to be distinguished is a local property. Our main example at the moment is $X = \mathbb{C}$, where the distinguished functions are the holomorphic functions.

A **geometric space** is a pair $(X, \mathcal{O})$ consisting of a topological space and a geometric structure.

1.2 Definition. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is a continuous map $f : X \to Y$ with the following additional property. If $V \subset Y$ is open and $g \in \mathcal{O}_Y(V)$ then $g \circ f$ is contained in $\mathcal{O}_X(f^{-1}(V))$.* DmG

Quite trivial facts are:

The composition of two morphisms is a morphism.
The identical map $(X, \mathcal{O}) \to (X, \mathcal{O})$ is a morphism.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is called an isomorphism if f is topological and if $f^{-1} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is also a morphism. This means that the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(f(U))$ are naturally isomorphic.

Let $U \subset X$ be an open subset of a geometric space $(X, \mathcal{O})$. We can define the restricted geometric structure $\mathcal{O}|U$ by

$$\mathcal{O}|U(V) := \mathcal{O}(V) \qquad (V \subset U \text{ open}).$$

It is clear that the natural embedding $i : (U, \mathcal{O}_X|U) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism and moreover that a map $f : Y \to U$ from a geometric space $(Y, \mathcal{O}_Y)$ into U is a morphism if and only if $i \circ f$ is a morphism.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called an **open embedding**, if it is the composition of an isomorphism $(X, \mathcal{O}_X) \to (U, \mathcal{O}_Y|U)$, $U \subset Y$ open, and the natural injection.

2. The notion of a Riemann surface

We equip $\mathbb{C}$ and more generally an open subset V with the sheaf of holomorphic functions. The geometric space obtained in this way is denoted by $(V, \mathcal{O}_V)$. A Riemann surface is a geometric space which is locally isomorphic to such a space:

2.1 Definition. *A Riemann surface is a geometric space $(X, \mathcal{O}_X)$, such thaat for every point there exists an open neighborhood U and an open subset $V \subset \mathbb{C}$ such that the geometric spaces $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_V)$ are isomorphic geometric spaces.* DRF

Of course $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ is a Riemann surface. An open subspace of a Riemann surface (equipped with the induced geometric structure) is a Riemann surface. Especially every open subset $U \subset \mathbb{C}$ carries a natural structure of a Riemann surface. If U, V are two open subsets of $\mathbb{C}$ then for a map $f : U \to V$ the following two conditions are equivalent:

a) The map f is analytic in the sense of complex analysis.
b) The map f defines a morphism of geometric spaces $f : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$.

This allows us to define:

2.2 Definition. *A map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between Riemann surfaces is called holomorphic if it is a morphism of geometric spaces.* Dha

A biholomorphic map between Riemann surfaces of course is a bijective map with is analytic in both directions.

A (topological) chart on a Riemann surface X is a topological map from an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. The chart is called analytic if it is moreover biholomorphic, i.e. an isomorphism of geometric spaces $(U, \mathcal{O}_X|U) \to (V, \mathcal{O}_V)$.

Let $\varphi : U \to V$ and $\psi : U' \to V'$ be two charts on X. then we can consider the topological map

$$\gamma := \psi \circ \varphi^{-1} : \varphi(U \cap U') \to \psi(U \cap U').$$

The notation "$\psi \circ \varphi^{-1}$" is not in accordance with the strong rules of set theory, but we allow this and related notations when it is clear from the context what is meant.

The sets $\varphi(U \cap U')$ and $\psi(U \cap U')$ are open subsets of $\mathbb{C}$. If the charts φ, ψ are analytic then the chart change map γ is biholomorphic.

A set of charts $\varphi : U \to V$ is called an *atlas* of X, if the domains of definition U cover X. The set of all analytic charts is an atlas.

Riemann surfaces via charts

Assume that a topological space X and a set $\mathcal{A}$ of two dimensional charts (topological maps from open subsets of X to open subsets of $\mathbb{C}$) is given. We assume that $\mathcal{A}$ is an atlas. We also assume that all chart changes $\gamma = \psi \circ \varphi^{-1}$ $(\varphi, \psi \subset \mathcal{A})$ are biholomorphic. We call than $\mathcal{A}$ an analytic atlas.

2.3 Remark. *Let $\mathcal{A}$ be an analytic atlas on X. Then there exists a unique structure as Riemann surface $(X, \mathcal{O}_X)$ such that all elements of $\mathcal{A}$ are analytic charts with respect to this structure* DmK

It should be clear how $\mathcal{O}_X$ has to be defined: A function $f : U \to X$ for open $U \subset X$ belongs to $\mathcal{O}_X(U)$ is and only if for every chart $\varphi : U_\varphi \to V_\varphi$ the function

$$f_\varphi = f \circ \varphi^{-1} : \varphi(U \cap U_\varphi) \longrightarrow \mathbb{C}$$

is analytic in the usual sense.

The atlas $\mathcal{A}$ then is part of the atlas of all analytic charts of $(X, \mathcal{O}_X)$, but it can be smaller. The atlas of all analytic charts is the unique maximal analytic atlas which contains $\mathcal{A}$.

3. Meromorphic functions

We recall the topology of the Riemann sphere A subset $U \subset \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is open if $U \cap \mathbb{C}$ is open $\mathbb{C}$ ist ind if in the case $\infty \in U$ there exists $C > 0$ with the property

$$z \in \mathbb{C}, \quad |z| > C \Longrightarrow z \in U.$$

Obviously $\bar{\mathbb{C}}$ is a compact space The subset $\mathbb{C}$ is open and the induced topology is the usual one.

Let $U \subset \bar{\mathbb{C}}$ an open set and $f : U \to \mathbb{C}$ a function. We assume that $\infty \in U$. Then $f(1/z)$ is defined in an open neighborhood of 0. We call f analytic at ∞ if $f(1/z)$ is analytic in an open neighborhood of zero. For any open set we define $\mathcal{O}_{\bar{\mathbb{C}}}(U)$ to be the set of all functions with the following properties:

a) The restriction of f to $U \cap \mathbb{C}$ is analytic in the usual sense.
b) When $\infty \in U$ then f is analytic at ∞.

It is easy to see that $(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}})$ is a Riemann surface. We describe two analytic charts which cover $\bar{\mathbb{C}}$. They can be used to introduce $\bar{\mathbb{C}}$ as Riemann surface via charts.

1) $$\bar{\mathbb{C}} - \{\infty\} = \mathbb{C} \xrightarrow{\mathrm{id}_{\mathbb{C}}} \mathbb{C},$$

2) $$\bar{\mathbb{C}} - \{0\} \longrightarrow \mathbb{C}, \qquad z \longmapsto 1/z \quad (1/\infty = 0),$$

The chart change map is

$$\mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}, \qquad z \longmapsto 1/z.$$

We consider now holomorphic maps $f : X \to \bar{\mathbb{C}}$ of an arbitrary Riemann surface into the Riemann sphere.

3.1 Definition. *A meromorphic function f on a Riemann surface X is an analytic map* defMER

$$f : X \longrightarrow \bar{\mathbb{C}},$$

such that$f^{-1}(\infty)$ is a discrete subset of X.

The constant function $f(z) = \infty$ is an analytic map but not meromorphic. Let $S \subset X$ be a discrete subset and $f : X \to \mathbb{C}$ a holomorphic function. It may happen that f extends to a meromorphic function on X. Then we call the points in S unessential singularities. Since the extension of f to X is unique we will use the same letter for it.

We denote by $\mathcal{M}(X)$ the set of all meromorphic functions on X. Let $f, g \in \mathcal{M}(X)$ be two meromorphic functions on X and denote by S the union of the points were f or g has the value ∞. Then we can define on $X - S$ the analytic functions $f + g$ and $f \cdot g$. The points of S are inessential singularities. Hence we have defined

$$f + g, \quad fg \in \mathcal{M}(X).$$

It follows that $\mathcal{M}(X)$ is a ring.

If f is a meromorphic function with discrete zeros set then $1/f$ is an analytic function on the complement of this set. Since the singularities are inessential it extends as meromorphic function on X.

When $f \in \mathcal{O}_X(X)$ is an analytic function the composition with the natural inclusion $\mathbb{C} \hookrightarrow \bar{\mathbb{C}}$ is a meromorphic finction. We identify f with this meromorphic function. This means that we can consider $\mathcal{O}_X(X)$ as a subring of $\mathcal{M}(X)$.

We want to show that $\mathcal{M}(X)$ is a field, wenn X is connected. For this we need a a variant of the principle of analytic continuation.

3.2 Lemma. *Let $f, g : X \longrightarrow Y$ be two analytic maps of a connected Riemann surface X into a Riemann surface Y. Assume that there exists a subset $A \subset X$ which is not discrete and such that f and g agree on A. Then $f = g$.* MeKor

Corollary. *Let $f : X \to Y$, X connected, be a non-constant analytic map than $f^{-1}(y)$ is discrete for every $y \in Y$. $b \in Y$ diskret in X.*

Corollary. *The set $\mathcal{M}(X)$ of all meromorphic functions on a connected Riemann surface is a field.*

Zum Beweis von 3.2 *Proof.* Consider the set of all cumulation points of the set $\{x \in X;\quad f(x) = g(x)\}$. It is sufficient to show that this set is open and closed. Since this is a statement of local nature, one can take analytic charts and reduce the statement to the case where X and Y are open subsets of $\mathbb{C}$. Now one can use the standard principle of analytic continuation.

4. Ramification points

An analytic map $f : X \to Y$ of Riemann surfaces is called locally biholomorphic at a point $a \in X$, if f maps some open neighborhood of a biholomorphic to an open neighborhood of $f(a)$. If this is not the case, a is called a ramification point.

4.1 Remark. *Let $f : X \to Y$ be a non constant holomorphic map of connected Riemann surfaces. The set of ramification points is discrete in X.* SRd

Proof. Taking charts one can assume that X and Y are open subsets of $\mathbb{C}$. The ramifictation points then are the zeros of the derivative of f. □

We recall a result of complex calculus. Let $f : U \to \mathbb{C}$ be a non-constant analytic function on an open connected neighborhood of 0 with the property $f(0) = 0$. There exists a small open neighborhood $0 \in U \in V$ and an analytic function $h : V \to \mathbb{C}$ with the properties

$$f(z) = h(z)^n, \qquad h'(0) \neq 0.$$

If V is taken small enough then h maps V biholomorphic onto an open neighborhood of 0. (The number n is the zero-order if f.)

We want to reformulate this result for Riemann surfaces. For sake of convenience wi will use the following notation:

A disc around $a \in X$ on a Riemann surface X is a biholomorphic map (=analytic chart)

$$\varphi : U \xrightarrow{\sim} E, \qquad \varphi(0) = 0.$$

Here

$$E := \left\{ q \in \mathbb{C};\ |q| < 1 \right\}$$

denotes the unit disc.

If b is a point of U also say that the disc contains a. If U is a subset of a subset $A \subset X$, then we say that the disc is contained in A. (This is not quite correct since a disc is map and not only the set U.)

If a is a point of a Riemann surface then of course there exists a disc around a. One simply takes an analytic chart $\varphi : U \to V$, $a \in U$. Then one replaces V be a small disc around $f(a)$ and U by the inverse image. Trivially there exists a biholomorphic map of an arbitrary disc to the unit disk, such that the center goes to the center. One can a say a little more:

Around an arbitrary point a of a Riemann surface there exist arbitrary small discs.

This means of course that in a given neighborhood one can find a disc which is contained in it.

4.2 Remark. *Let $f : X \to Y$ be a non-constant map of connected Riemann surfaces and $a \in X$ a given point. There exists discs $\psi : V \to E$ around $f(a) \in Y$ and $\varphi : U \to E$ around a such that the diagram* LBda

$$\begin{array}{ccc} U & \xrightarrow{\ \varphi\ } & E \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ V & \xrightarrow{\ \psi\ } & E \end{array} \qquad\qquad \begin{array}{c} q \\ \big\downarrow \\ q^n \end{array}$$

commutes for some natural number n.

A simple way to express 4.2 is:

Analytic maps of Riemann surfaces look locally like "$q \mapsto q^n$".

The point is a ramification point if and only if $n > 1$.

Proper maps

For *proper* analytic maps $f : X \to Y$ there are better results. We will use in the following frequently the following trivial fact: Let $V \subset Y$ an open subset and $U = f^{-1}(V)$ its (full!) inverse image. Then $f : U \to V$ is proper too.

The basic fact which we have to use is the following purely topological result.

4.3 Proposition. *Let $f : X \to E^\bullet$ be a locally topological proper map of a connected Hausdorff space into the punctured unit disc $E^\bullet = E - \{0\}$. Then there exists a topological map $\sigma : X \to E^\bullet$ such that the diagram* UuvE

$$\begin{array}{ccccc} X & & \xrightarrow{\ \sigma\ } & & E^\bullet \\ & {\scriptstyle f}\searrow & & \swarrow & \\ & & E^\bullet & & \end{array} \qquad\qquad q \mapsto q^n$$

commutes.

We don't give a proof here. But we point out as a consequence.

4.4 Corollary of 4.3. *Let in addition X be Riemann surface and f analytic. Then The map σ is biholomorphic.* UuvF

We treat a topological application of 4.3. We take (in the notation of 4.3 a symbol a which is not contained in X. The we define $\bar{X} = X \cup \{a\}$. Then we extend the map σ to the map

$$\bar{\sigma} : X \longrightarrow E, \quad \bar{\sigma}(a) = 0.$$

This is bijective map. Then we equip $\bar{X}$ with the unique topology such that this map is topological. Then X is an open subspace of $\bar{X}$. We also extend f to a map

$$\bar{f} : \bar{X} \longrightarrow E, \qquad f(a) = 0.$$

This map still is continuous, even proper but not locally topological.

This trivial construction admits an important extension:

4.5 Proposition. *Let $\bar{Y}$ be a surface and $Y \subset \bar{Y}$ an open subset such that $T := \bar{Y} - Y$ is a finite set. Assume that $f : X \to Y$ is a locally topological proper map of a Hausdorff space X into Y. Then there exists a surface $\bar{X}$ with the following property:* Kea

a) *X is an open subset of $\bar{X}$ and the topology of Y is the induced topology.*
b) *$T := \bar{X} - X$ is a finite set. The map f extends to a continuous map*

$$\bar{f} : \bar{X} \longrightarrow \bar{Y}.$$

c) *For every point $a \in T$ there exist neighborhoods $a \in U \subset \bar{X}$ and $\in V \subset \bar{Y}$ and topological maps $U \to E$ and $V \to E$ such that the diagram*

$$\begin{array}{ccccc} U & \longrightarrow & E & \qquad & q \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & E & & q^n \end{array}$$

commutes.

Additional Remark. *When $\bar{Y}$ is compact then so is $\bar{X}$.*

Proof. For each $b \in T$ we choose an open neighborhood $V(b)$ and a topological map $V(b) \to E$, $b \mapsto 0$. We can assume that the $V(b)$ for different b are disjoint. Next we take the inverse image $f^{-1}(V(b))$ in X. This needs not to be connected. Hence we consider the connected components. From the properness of f follows that there are only finitely man connected components. Let U be one of them. The map

$$U \to V(b) - \{b\} \cong E^{\bullet}$$

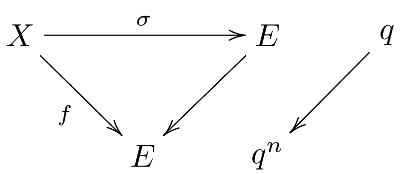
still is proper. To this situation we apply the above construction adding one extra point to U. To be precise we take for each point $b \in T$ and for each connected component of $f^{-1}(V(b))$ an extra symbol $a_{b,U}$. Then we add to U this symbol to get $\bar{U} = U \cup \{a_{b,U}\}$. Then we consider

$$\bar{X} = X \cup \bigcup_{b \in T,\ U} \{a_{b,U}\}.$$

Then $\bar{U}$ is a subset of $\bar{X}$. It is clear how to extend f to a map $\bar{f} : \bar{X} \to \bar{Y}$. It is also clear hoe to define the topology on $\bar{X}$. A subset is called open if the intersection with X and all U is open. □

If $f : U \to \mathbb{C}$ is a non-constant function on a connected open subset $U \subset \mathbb{C}$. Then for $a \in U$ the zero order of $f(z) - f(a)$ at a is called the multiplicity of f at a. If $f : X \to Y$ is a non constant holomorphic map of Riemann surfaces, the the multiplicity of f in a point $a \in X$ can defined in an obvious way via charts. It is the number n of 4.2. A weak form of 4.3 shows:

4.6 Lemma. *Let $f : X \to E$, $a \mapsto 0$, be a holomorphic and proper map of a connected Riemann surface, which is locally biholomorphic outside a. Then there exists a biholomorphic map $\sigma : X \to E$ such that the diagram* UuvF



commutes.

Proof. From 4.3 follows the existence of a biholomorphic map $f : X - \{\} \to E^{\bullet}$. We extend by $f(a) = 0$. The Riemann extension theorem shows that this map is still holomorphic. □

From 4.6 and the above discussion we deduce:

4.7 Proposition. *Let $f : X \to Y$ be a proper non-constant holomorphic map of connected Riemann surfaces. Denote for $b \in Y$ by n the number of all $a \in X$ with $f(a) = b$ counted with multiplicity. The number n is independent of b. Especially f is surjective.* EasU

We denote the number n the *covering degree* of f.

5. Examples of Riemann surfaces

The only Riemann surfaces which we introduced so far is the Riemann sphere and its open subsets. We give some more interesting example but keep short since they are not needed for the development of the general theory.

Tori

Let $L \subset \mathbb{C}$ be a lattice, i.e. $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where ω_1, ω is an $\mathbb{R}$-basis of $\mathbb{C}$. The quotient $X_L = \mathbb{C}/L$ carries a natural structure as Riemann surface. The topology is the quotient topology. A function $f : U \to \mathbb{C}$ on an open subset $U \subset X_L$ is called holomorphic if and only if the composition with the natural projection $p : \mathbb{C} \to X_L$ is a holomorphic function in the usual sense on $p^{-1}(U)$. The natural projection p is holomorphic and even more locally biholomorphic. It follows the meromorphic functions $f : X_L \to \bar{\mathbb{C}}$ are in 1-1-correspondence to the elliptic functions $F = f \circ p$ (meromorphic functions on $\mathbb{C}$ which are periodic with respect to L.

It can be shown that two tori X_{L_1} and X_{L_2} are biholomorphic equivalent if and only if there exists a complex number $a \in \mathbb{C}$ such that $L_2 = aL_1$. Since this is usually not the case and since two tori are always topologically equivalent, we see that topologically equivalent Riemann surfaces are usually not biholomorphic equivalent.

Concrete Riemann surfaces

A function element (a, P) is point $a \in \mathbb{C}$ together with an element $P \in \mathcal{O}_a$. We can think of P as power series with center a and positive convergence radius. Let $\alpha : [0, 1] \to \mathbb{C}$ a curve with starting point $a = \alpha(0)$. Assume that a function element (a, P) is given. We say that (a, P) admits analytic continuation along α if there exist for every $t \in [0, 1]$ a function element $(\alpha(t), P_t)$ such that the following condition hold:

a) $(a, P) = (\alpha(0), P_0)$.
b) Let $t \in [0, 1]$. If $t' \in [0, 1]$ is close enough to t than the (open) convergence discs of P_t and $P_{t'}$ have non empty intersection and both functions agree in the intersection.

It is easy to see that such an analytic continuation is unique. Two function elements (a, P) and (b, Q) are called equivalent if there exists a path α with $\alpha(0) = a$ and $\alpha(1) = b$ such that (a, P) can be continued analytically to b with the end-element (q, Q).

The basic truth is that (b, Q) depends on the choice of the curve α. As an example on considers the function element $(1, \sqrt{z})$, where $\sqrt{z}$ is defined by the principal part of the logarithm. This function element can be continued to -1. But if one takes the continuation along the half circle in the upper half plane one gets a different result as if one takes the half circle in the low half plane. The two continuations differ by a sign.

The idea of the Riemann surface is to count the point -1 twice. This means one takes a two-fold covering of $\mathbb{C}^*$. On this covering $\sqrt{z}$ can be defined as an unambiguous.

To define this in mathematical rigorous way, one starts with an analytic function $f : D \subset \mathbb{C}$ on some connected open subset of $\mathbb{C}$. The elements (a, f_a) of course all are equivalent. Now we introduce the set $\mathcal{R} = \mathcal{R}(f)$ od all functions element which are equivalent to the elements (a, f_a). So $\mathcal{R}$ contains all possible analytic continuations of f.

We introduce a topology on $\mathcal{R}$. Let $(a, P) \in \mathcal{R}$. Consider a positive number ε which is smaller than the convergence radius of r. Then we define

$$U_\varepsilon(a, P) := \big\{\, (b, Q); \quad b \in U_\varepsilon(a),\ Q \text{ is the germ of } P \text{ in } b \,\big\}.$$

A subset $U \subset \mathcal{R}$ is called open, if for every point $a \in U$ there exists a small $\varepsilon > 0$ such that

$$U_\varepsilon(a, P) \subset U.$$

It is clear that this is a (Hausdorff) topology on $\mathcal{R}$. The natural projection

$$p : \mathcal{R} \longrightarrow \mathbb{C}, \quad (a, P) \longmapsto a,$$

is continuous and moreover the restriction

$$U_\varepsilon(a, P) \longrightarrow U_\varepsilon(a) \qquad (\varepsilon \text{ small enough})$$

is topological. Hence we see that the map $\mathcal{R} \to \mathbb{C}$ is locally topological. This is enough information to equip $\mathcal{R}$ with a structure as Riemann surface.

5.1 Remark. *Let Y be a Riemann surface and $f : X \to Y$ a locally topological map of a Hausdorff space X into Y. Then X carries a unique structure as Riemann surface such that f is holomorphic.* UesR

Proof. One defines $\mathcal{O}_X(U)$ to be the set of all function $f : U \to \mathbb{C}$ such that for all $U_\varepsilon(a, P) \subset$ the composition

$$U_\varepsilon(a) \xrightarrow{\sim} U_\varepsilon(a, P) \xrightarrow{f} \mathbb{C}$$

id analytic in the usual sense. It is easy to verify the demanded properties. □

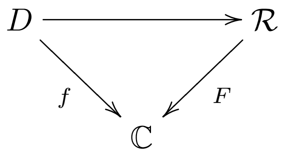
Besides the projection $\mathcal{R} \to \mathbb{C}$ on can consider the map

$$F : \mathcal{R} \longrightarrow \mathbb{C}, \quad F(a, P) = P(a).$$

This function is of course analytic. There is a natural map

$$D \longrightarrow \mathcal{R}, \quad a \longmapsto (a, f_a),$$

which obviously is an open imbedding. The basic fact is the commutative diagram



shows F to be an extension of the original function f. This extension includes all possible analytic continuations of f. In a naiv sense they are a multi-valued function. The idea of the Riemann surface $\mathcal{R}$ is to obtain a single-valued function on a covering of $\mathbb{C}$.

The Riemann surface of an algebraic function

Let $P(z, w)$ by a polynomial of two variables. We assume in the following that P is irreducible. This means that P cannot be written as product of two non-constant polynomials. We also assume that P depends on both variables properly. The affine curve associated to P is defined as

$$\mathcal{N} := \{ (z, w) \in \mathbb{C} \times \mathbb{C}; \quad P(w, z) = 0 \}.$$

5.2 Lemma. *The fibres of the map $p : \mathcal{N} \to \mathbb{C}$, $p(z, w) = z$, are finite* Palf

This is an easy application of the fact, that a non-zero polynomial in one variable has only finitely many zeros. □

5.3 Proposition. *There are finite subsets $S \subset \mathbb{C}$ and $T = p^{-1}$ such that* Palt

$$p : \mathcal{N} - T \longrightarrow \mathbb{C} - T$$

is proper and locally topological.

We just describe the set S: A point $a \in \mathbb{C}$ is contained in S if one the following conditions is satisfied:

a) There is a b such that P and $\partial P / \partial w$ both vanish at (a, b).
b) Write P in the form

$$P(z, w) = a_0(z) + \cdots + a_n(z) w^n, \quad a_n \neq 0.$$

Then $a_n(s) = 0$.
We don't give more details and mention just that b) implies that $p : \mathcal{N} - T \longrightarrow \mathbb{C} - T$ is locally topological. (One has to use the theorem of implicit functions.) That p is proper uses a). □

Now we imbed $\mathbb{C} - S$ into the Riemann sphere $\bar{\mathbb{C}}$ and apply 4.5.

5.4 Theorem. *There exists a compact Riemann surface X which contains $\mathcal{N}-T$ as open sub-surface (with the induced topology and the restricted geometric structure). The complement is a finite set. The map $p:\mathcal{N}-T\longrightarrow\mathbb{C}-T$ extends to a meromorphic function* Rfaf

$$p:X\longrightarrow\bar{\mathbb{C}}.$$

We also mention a result, which often is clear in concrete situations but needs a proof in general.

5.5 Theorem. *The compact Riemann surface X attached to an algebraic function is connected. The second projection $q(z,w)=w$ extends also to a meromorphic function* Rfiz

$$q:X\longrightarrow\bar{\mathbb{C}}.$$

We end this section with a special example in which the above results can be proved less more easily. Let $Q(z)$ be a non-constant polynomial in 1 variable without multiple zeros and take $P(z,w)=w^2-Q(z)$. It is easy to show that P and $\partial P/\partial w$ have now common zeros. It follows that $\mathcal{N}$ is already a surface (an imbedded manifold in the sense of analyis). It carries a structure as Riemann surface sich that $p:\mathcal{N}\to\mathbb{C}$ is proper and holomorphic. The covering degree is two. The compactifictation $\mathcal{N}\subset X$ needs one ore two additional points. If it is one the map p is locally of the form $z\mapsto z^2$. If there are two the map is locally biholomorphic at both. Later we will see that the compactifaction by one (resp. two) points depends on whether the degree of Q is odd (resp. even).

Recall that we have two projections $p,q:X\to\bar{\mathbb{C}}$ which are induced by $p(z,w)=z$ and $q(z,w)=w$. The first projections plays of the role of the natural "coordinate" on X. Hence we write simply z instead of q. The second projection q describes the solution of the equation $w^2=Q(z)$. Hence we write simply $\sqrt{Q(z)}$ for q. We see that the a priori "double valued function $\sqrt{Q(z)}$ appears as usual single-valued function q on a two-fold covering of $\bar{\mathbb{C}}$.

Finally we mention that an alternative construction of the Riemann surface of an algebraic can be given by means of the concrete Riemann surface of a local solution $w=w(z)$ of the equation $P(z,w)=0$.

6. Differential forms

Let $U\subset\mathbb{C}$ be an open subset. We consider the space $A^p(U)$ $(p=0,1,2)$ of complex valued $\mathcal{C}^\infty$-differential forms,

$$A^0(U)=\mathcal{C}^\infty(U),\quad A^1(U)=\mathcal{C}^\infty(U)\times\mathcal{C}^\infty(U),\quad A^2(U)=\mathcal{C}^\infty(U).$$

As usually we write the elements of $A^1(U)$ in the form $f dx + g dy$ and the elements of $A^2(U)$ in the form $f dx \wedge dy$. We recall the operators

$$A^1(X) \times A^1(X) \longrightarrow A^2(X),$$
$$(f_1 dx + g_1 dy) \wedge (f_2 dx + g_2 dy) = (f_1 g_2 - f_2 g_1) dx \wedge dy,$$

(exterior product) and the exterior derivatives

$$d : A^0(U) \longrightarrow A^1(U), \quad df = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

and

$$d : A^1(U) \longrightarrow A^2(U), \quad d(f dy + g dy) = \Big(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big)\, dx \wedge dy.$$

One has $d \circ d = 0$. The Lemma of Poincaré states that

$$0 \longrightarrow \mathbb{C} \longrightarrow A^0(U) \longrightarrow A^1(U) \longrightarrow A^2(U) \longrightarrow 0$$

is exact for convex U. We use the notations

$$dz = dx + \mathrm{i} dy, \qquad d\bar z = dx - \mathrm{i} dy$$

and

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big).$$

We can write a one-form also as $f dz + g d\bar z$. The operator d can be rewritten as

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z \quad \text{and} \quad d(f dz + g d\bar z) = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz + \frac{\partial g}{\partial z} dz \wedge d\bar z$$

Notice

$$dz \wedge d\bar z = -d\bar z \wedge dz = 2\mathrm{i} dx \wedge dy.$$

We obtain a splitting

$$A^1(U) = A^{1,0}(U) \oplus A^{0,1}(U)$$

with $A^{1,0}(U) = \mathcal{C}^\infty(U) dz$ and $A^{0,1}(U) = \mathcal{C}^\infty(U) d\bar z$. We also define $A^{1,1}(U) := A^2(U)$. A one-differential is called holomorphic, if it is of the form $f dz$ with holomorphic f. From the Cauchy-Riemann equations follows

$$d(f(z)dz) = f'(z) dz \quad \text{for holomorphic } f.$$

The set of all holomorphic one-forms is denoted by $\Omega(U)$. It is a subspace of $A^{1,0}(U)$. We also introduce the operators

$$\bar\partial : A^0(U) \longrightarrow A^{0,1}(U), \qquad \bar\partial(f) := \frac{\partial f}{\partial \bar z}$$

and

$$\bar\partial : A^{1,0}(U) \longrightarrow A^{1,1}(U), \qquad \bar\partial(f dz) := \frac{\partial f}{\partial \bar z} d\bar z \wedge dz.$$

The latter coincides with d. The Lemma of Dolbeault states:

6.1 Dolbeault lemma. *For a disc U the sequence* DL

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow A^0(U) \xrightarrow{\bar\partial} A^{0,1}(U) \longrightarrow 0$$

is exact or equivalently

$$0 \longrightarrow \Omega(U) \longrightarrow A^{1,0}(U) \xrightarrow{\bar\partial} A^{1,1}(U) \longrightarrow 0$$

is exact.

Transformation of differential forms

Let $\gamma = \gamma_1 + \mathrm{i}\gamma_2 : U \to V$ be a $\mathcal{C}^\infty$-map of open subsets $U, V \subset \mathbb{C}$. One defines the pull-back

$$\gamma^* : A^p(V) \longrightarrow A^p(U)$$

as follows:

a) for 0-forms: $\gamma^*(f) = f \circ \gamma$,

b) for 1 forms: $\gamma^*(f dx + g dy) = (f \circ \gamma)\, d\gamma_1 + (f \circ \gamma)\, d\gamma_2$,

c) for 2-forms: $\gamma^*(f dx \wedge dy) = (f \circ \gamma) \left(\frac{\partial\gamma_1}{\partial x}\frac{\partial\gamma_2}{\partial y} - \frac{\partial\gamma_1}{\partial y}\frac{\partial\gamma_2}{\partial x} \right) dx \wedge dy.$

The chain rule says: $\gamma_1^* \circ \gamma_2^* = (\gamma_2 \circ \gamma_1)^*$. Another consequence of the chain rule is that the derivative d is compatible with transformation:

$$\gamma^*(d\omega) = d(\gamma^*\omega).$$

When $w = \gamma(z)$ is holomorphic and $f(w)dw$ a holomorphic 1-form, then the transformation formula simplifies because of the Cauchy-Riemann equations as follows:

$$\boxed{\gamma^*(f(w)dw) = f(\gamma z)\gamma'(z)dz \qquad (f \text{ holomorphic}).}$$

Finally we mention

$$\gamma^*(\omega \wedge \omega') = (\gamma^*\omega) \wedge (\gamma^*\omega')$$

which is easy to verify.

Differential forms on Riemann surfaces

By a p-form on a Riemann surface one understands a family of p-forms $\omega_\varphi \in A^p(V_\varphi)$, where $\varphi : U_\varphi \to V_\varphi$ runs through all analytic charts, such that the following condition is satisfied:

If φ, ψ are two analytic charts and $\gamma = \psi \circ \varphi^{-1}$ is the coordinate change map then

$$\omega_\psi = \gamma^*\omega_\varphi$$

holds on the open subsets where γ is defined. If one has a family, which is only defined for all φ form an atlas of analytic charts then this family extends uniquely to the atlas of all analytic charts. All what one has to use to us is the chain rule $\gamma_1^* \circ \gamma_2^* = (\gamma_2 \circ \gamma_1)^*$ and the fact that transformation and derivatives are compatible with restriction to smaller open subsets.

A zero form especially is a family of functions $f_\varphi : V_\varphi \to \mathbb{C}$ such that the transported functions in U_φ coincide in the intersections. Hence they glue to a function $f : X \to \mathbb{C}$. We will identify f with the family (f_φ). We denote by $A^p(X)$ the space of p-forms. We have $A^0(X) = \mathcal{C}^\infty(X)$.

A 1-form is called holomorphic if all its components ω_φ are holomorphic. It is enough to demand this for an atlas of analytic charts, since the transform of a holomorphic 1-form by a holomorphic transformation is holomorphic again. (Here we make use of the fact that we have a Riemann surface and not only a differentiable surface in the sense of real analysis). We denote by $\Omega(X)$ the space of holomorphic 1-forms.

Another obvious property of *holomorphic* transformations is that it preserves the type $A^{0,1}$ and $A^{0,1}$. This also follows from the Cauchy Riemann differential equations. Hence we can define $A^{0,1}(X)$ and $A^{1,0}(X)$ for a Riemann surface componentwise. The operations d and ∂ commute with holomorphic transformations. Hence we can define operators

$$\begin{aligned} d &: \mathcal{C}^\infty(X) \longrightarrow A^1(X), & d &: A^1(X) \longrightarrow A^2(X), \\ \bar\partial &: \mathcal{C}^\infty(X) \longrightarrow A^{0,1}(X), & \bar\partial &: A^{1,0}(X) \longrightarrow A^2(X) \end{aligned}$$

also componentwise (for example $d(\omega)_\varphi := d(\omega_\varphi)$.)

Also the exterior product generalizes to Riemann surfaces via charts.

Sheaves of differential forms

If one attaches to each open subset U of a Riemann surface the various spaces of differential forms one obtains sheaves which we denote by $\mathcal{A}_X^p$, $\mathcal{A}_X^{0,1}$, $\mathcal{A}_X^{1,0}$, Ω_X. We also obtain sequences of sheaves:

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{A}_X^1 \longrightarrow \mathcal{A}_X^2 \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{A}_X^{0,1} \longrightarrow 0,$$
$$0 \longrightarrow \Omega_X \overset{\text{inclusion}}{\longrightarrow} \mathcal{A}_X^{1,0} \longrightarrow \mathcal{A}_X^2 \longrightarrow 0.$$

These are exact sequences of sheaves. Hence we obtain:

6.2 Theorem of de-Rham. *Let X be a connected Riemann surface. Then* TdRz

$$H^0(X,\mathbb{C}) = \mathbb{C}, \quad H^1(X,\mathbb{C}) = \frac{\operatorname{Kernel}(A^1(X) \longrightarrow A^2(X))}{\operatorname{Image}(\mathcal{C}^\infty(X) \longrightarrow A^1(X))}$$

$$H^2(X,\mathbb{C}) = \frac{A^2(X)}{\operatorname{Image}(A^1(X) \longrightarrow A^2(X))}, \quad H^q(X,\mathbb{C}) = 0 \textit{ for } q > 2$$

6.3 Theorem of Dolbeault. *Let X be a connected Riemann surface. Then* TdD

$$H^1(X,\mathcal{O}_X)=\frac{A^{0,1}(X)}{\mathrm{Image}(\mathcal{C}^\infty(X)\longrightarrow A^{0,1}(X))},\quad H^q(X,\mathcal{O}_X)=0 \textit{ for } q>1.$$

and

$$H^1(X,\Omega_X)=\frac{A^{2}(X)}{\mathrm{Image}(A^{0,1}(X)\longrightarrow A^{2}(X))},\quad H^q(X,\Omega_X)=0 \textit{ for } q>1.$$

When X is an open subset of $\mathbb{C}$, the formulae for $\mathcal{O}_X$ and Ω_X mean the same. This is not true for arbitrary X.

The Stokes formula

We explain shortly the integration of differential forms. If $\omega = f_1dx + f_2dy$ is a 1-form on an open set $V\subset\mathbb{C}$ and $\alpha:[0,1]\to U$ a smooth (=infinitly differentiable) curve, $\alpha=\alpha_1+\mathrm{i}\alpha_2$, then one defines

$$\int_\alpha \omega := \int_0^1 \big(f_1(\alpha(t))\alpha_1'(t)+f_2(\alpha(t))\alpha_2'(t)\big)\,dt.$$

When $\gamma:U\to V$ is a diffeomorphism then

$$\int_\alpha \omega = \int_{\alpha\circ\gamma}\gamma^*\omega.$$

This allows us to generalize this to Riemann surfaces: Let $\omega\in A^1(X)$ and $\alpha:[0,1]\to X$ a smooth curve. (It is clear how to define smoothness for α. One uses charts.) Then one can define $\int_\alpha\omega$. One divides α in pieces which lie in analytic charts. Then one uses the local formula. Because of the transformation invariance this is independent of the choice of the charts.

Similarly one defines the integral of a two-form ω. In the local case of an open subset $V\subset\mathbb{C}$ one takes for

$$\int_V f(z)dx\wedge dy$$

the usual 2-dimensional integral of f (if it exists). Again we need transformation invariance. If $\gamma:U\to V$ is a diffeomorphism then

$$\int_\alpha \omega = \int_{\alpha\circ\gamma}\gamma^*\omega$$

only holds if the (real) Jacobi determinant of γ is positive. When γ is biholomorphic, this is automatically the case, since the real Jacobi determinant of γ is $|\gamma'|^2$. This is an important fact because otherwise we could not integrate 2-forms on Riemann surfaces. We want to keep it in mind:

6.4 Proposition. *Riemann surfaces are* ***oriented*** *in the following sense: The chart change map of two analytic charts has positive real Jacobi determinant.* RsOr

If ω is a 2-form on X which vanishes outside a chart φ then we can define

$$\int_X \omega := \int_{V_\varphi} \omega_\varphi.$$

The general case is reduced to this one by means of a partition of unity. Now we can formulate the formula of Stokes:

Let $U \subset X$ an open subset of a Riemann surface with compact closure. Assume that the boundary of U is the union of a finite number of (the images of) closed double point free regular curves $\alpha_i : [0,1] \to X$, $1 \le i \le n$. Assume furthermore that U is on the left of these curves. Then for every one-form ω on X

$$\int_U d\omega = \sum_{i=1}^n \int_{\alpha_i} \omega.$$

We recall that a closed curve α is called double point free if $\alpha(t) = \alpha(t')$ if only for $t, t' \in \{0,1\}$. Regular means $\alpha'(t) \ne 0$ for all t in case of a plain curve $\alpha : [0,1] \to \mathbb{C}$. The general case is reduced to that one by means of charts. An open subset $U \subset \mathbb{C}$ is called to the left of a regular curve α if for every t the following condition is satisfied. Consider the two vectors of norm 1 which are orthogonal to the tangent vector $\alpha'(t)$. Call $\mathfrak{n}^+$ the one which goes to the right (this makes sense in the plane) and $\mathfrak{n}^-$ the other one. Then there exists $\varepsilon > 0$ such that for all $0 \le s \le \varepsilon$

$$\alpha(t) + s\mathfrak{n}^- \in U, \quad \alpha(t) + s\mathfrak{n}^+ \notin U.$$

This can be generalized to open subsets U in a Riemann surface by means of analytic charts. One has to use 6.4.

The residue theorem on compact Riemann surfaces

Recall that a holomorphic differential (=1-form) on a Riemann surface is a collection $f_\varphi(z)dz$, where f_φ is a holomorphic function $f_\varphi : V_\varphi \to \mathbb{C}$ for every analytic chart (an atlas of analytic charts is enough) such that for two charts φ, ψ with chart change map γ the transformation formula

$$f_\psi(\gamma z) = \gamma'(z) f_\varphi(z)$$

holds. Instead of holomorphic functions one can take meromorphic functions. This leads to the notion of a *meromorphic differential* on a Riemann surface.

We recall the following residue formula from complex calculus: Let $\gamma : U \to V$ be a biholomorphic map of open subsets $U, V \subset \mathbb{C}$. Let $f : V \to \mathbb{C}$ be a holomorhic function and $a \in U$. Then

$$\operatorname{Res}(f(w), \gamma(a)) = \operatorname{Res}(\gamma'(z) f(\gamma z), a).$$

It is an important fact that the factor $\gamma'(z)$ occurs. This means that it makes no sense to talk about residues of *meromorphic functions* on Riemann surfaces. But for a *meromorphic differential* ω the definition

$$\operatorname{Res}(\omega, a) := \operatorname{Res}(\omega_\varphi, \varphi(a)) \qquad (a \in U_\varphi)$$

makes sense since it is independent of the choice of a chart.

6.5 Residue theorem. *Let ω be a meromorphic differential on a compact Riemann surface. Then* RT

$$\sum_{a \in X} \operatorname{Res}(\omega; a) = 0.$$

The sum is of course a finite sum. For the proof we take for each pole a a small neighborhood $U(a)$ such that the closures of two different are disjoint and such that the boundaries are nice (take discs with respect to charts). Then we apply the Stokes formula to the set

$$U = X - \bigcup_a \overline{U(a)}.$$

Since ω is holomorphic we have $d\omega = 0$. It follows

$$\sum_a \int_{\partial U(a)} \omega = 0.$$

This proves 6.5. □

If f is a non-vanishing meromorphic function on a connected Riemann surface, then we can consider the meromorphic differential df/f. From the formula

$$\operatorname{Res}\Big(\frac{df}{f}; a\Big) = \operatorname{Ord}(f, a)$$

follows:

6.6 Remark. *Let f be a non-constant meromorphic function on a connected compact Riemann surface. The number of poles and zeros —counted with multiplicity— is the same.* ANgP

This remark can also be proved without residue theorem. It is a special case of 4.7.

6.7 Remark. *Every holomorphic function on a connected compact Riemann surface is constant.* Hfcc

This follows from 6.6 but it follows also already form the maximum principle of complex calculus.

7. Integration of closed forms and homotopy

A differential ω is called closed if $d\omega = 0$ and total if it is of the form $\omega = df$. From the main theorem of calculus follows the formula

$$\int_\alpha df = \alpha(1) - \alpha(0)$$

for every smooth curve α. In the following it will be useful to weaken the smoothness of a curve. First of all the curve integral $\int_\alpha \omega$ can be generalized to piecewise smooth curves in an obvious way. But for *closed* differentials it is possible to extend the integral to arbitray continuous curves as follows. What we have to use is that closed forms are locally total by the Poincarè lemma. Hence every point admits a neighborhood U, such that $\int_\alpha \omega = 0$ for every closed curve in U when ω is closed. Let now be $\alpha : [0,1] \to X$ be an arbitrary (continuous) curve. By a compactness argument we can find a partition $0 = a_0 < \cdots < a_n = 1$ and open subsets $U_1, \dots, U_n$ such that

$$\alpha([a_{i-1}, a_i]) \subset U_i \qquad (1 \le i \le n)$$

and such that ω is total in U_i. Then we combine $\alpha(a_{i-1})$ and $\alpha(a_i)$ by some smooth curve inside U_i. This defines a new piecewise smooth curve β with the same origin and end as α. It is easy to see that $\int_\beta \omega$ is independent of the choice of β. (Here we use that ω is total in U_i.) Hence we can define

$$\int_\alpha \omega := \int_\beta \omega.$$

Let be $Q \subset \mathbb{C}$ be a compact rectangle, parallel to the axes and let $H : Q \to X$ be a continuous map. There can be defined in an obvious way a closed curve α which runs through the image of the boundary of Q. We claim

$$\int_\alpha \omega = 0 \qquad (\omega \text{ closed}).$$

The proof is very simple. One divides Q into small rectangles such that the image of each of them is contained in an open subset on which ω is total. Then the corresponding integral for the small rectangles are zero. Summing up all the integrals over the small rectangles on gets obviously the integral over the originale rectangle. We formulate a special case of this:

7.1 Definition. *Two curves $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ with the* DHO
property $\alpha(0) = \beta)0)$ and $\alpha(1) = \beta(1)$ are called ***homotopic****, if there exists a continuous map $H : [0,1] \times [0,1] \to X$, such that*

$$H(t,0) = \alpha(t), \quad H(t,1) = \beta(t)$$

and

$$H(0,s) = \alpha(0), \quad H(1,s) = \alpha(1).$$

The above observation shows:

7.2 Proposition. *Let α, β be two homotopic curves. Then* PCis

$$\int_\alpha \omega = \int_\beta \omega$$

for all closed differentials ω.

In the special case where ω is holomorphic, this is called the homotopy version of the Cauchy integral theorem.

The fundamental group

We fix a point $a \in X$ and consider now closed curves, which start from a and end in a. Homotopy defines an equivalence relation on this set. The set of all homotopy classes is denoted by $\pi_1(X, a)$. It is easy to show that composition of curves defines a well-defined product in $\pi_1(X, a)$. It is not difficult to show that this product makes $\pi_1(X, a)$ to a group. we will not give the details here. This group is called the fundamental group of (X, a).

7.3 Remark. *Let ω be a closed differential on X. Then integration defines a homomorphism* RcdH

$$\pi_1(X, a) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \int_\alpha \omega.$$

Chapter VI. The Riemann Roch theorem

1. Generalities about vector bundles

In the following we assume that the considered Riemann surfaces are connected.

1.1 Definition. *A **vector bundle** on a Riemann surface X is a locally free $\mathcal{O}_X$-module, i.e. a $\mathcal{O}_X$ module such that every point admits an open neighborhood such that $\mathcal{M}|U$ is isomorphic to $(\mathcal{O}_X|U)^n$ for some integer $n \ge 0$. If n can be taken to be 1, then $\mathcal{M}$ is called a **line bundle**.* DVb

When n is the same for all U, for example when X is connected, we call n then the rank of $\mathcal{M}$. Hence line bundles are vector bundles of rank 1.

We give a basic example for a line bundle. A divisor on a Riemann surface is a map $D : X \to \mathbb{Z}$ such that $D(a) = 0$ outside a discrete set. We are mainly interested in compact surfaces. Then this means that $D(a) = 0$ outside a finite set. We write D as formal linear combination

$$D = \sum_{a \in X} D(a)a.$$

The set of all divisors is an additive group $\operatorname{Div}(X)$ (componentwise addition). To every meromorphic function which is not zero on a component, we associate the so-called principal divisor (f).

$$(f)(a) = \operatorname{Ord}(f, a).$$

Because the formula $(fg) = (f) + (g)$ the set of all principal divisors is a subgroup $H(X) \subset \operatorname{Div}(X)$. The elements of the factor group $\operatorname{Div}(X)/H(X)$ are called divisor classes. Let f be a meromorphic function on sum open subset U. The notation

$$(f)(a) \ge n \qquad (a \in I,\ n \in \mathbb{Z})$$

means that $\operatorname{Ord}(f, a) \ge n$ if f doesn't vanish in a neighborhood of a. Hence $(f)(a) \ge n$ is always true if $f = 0$ in a neighborhood of a. Let D be a divisor on X. We write $(f) \ge D$ if $(f)(a) \ge D(a)$ for all a. Now we associate to a divisor D on X the following sheaf.

$$\mathcal{O}_D(U) := \big\{\, f : U \to \bar{\mathbb{C}} \text{ meromorphic}, \quad (f) \ge -D \,\big\}.$$

Clearly $\mathcal{O}_D$ is an $\mathcal{O}_X$-module. We claim that it is a line bundle: Let U is a sufficiently small neighborhood of a given point. Then there exists a meromorphic function g whose divisor on U coincides with $-D$. On simply has take U so small that it is in the domain of definition of a chart and that at most one $a \in U$ with $D(a) \neq 0$ exists. The map

$$\mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_D(U), \quad f \longmapsto gf,$$

then is an isomorphism. Since U can be replaced by smaller open subsets we obtain $\mathcal{O}_D|U \cong \mathcal{O}_X(U)$.

Two line bundles are called isomorphic, if they are isomorphic as $\mathcal{O}_X$-module.

1.2 Proposition. *Let X be a Riemann surface. Two divisors D and D' are in the same divisor class, if and only if the line bundles $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ are isomorphic.* DaLb

Proof. We use the following trivial algebraic fact: Every homomorphism $f : M \to N$ between R-modules which are isomorphic to R is given by multiplication with some element form R. From this follows easily: Let $\mathcal{O}_D \to \mathcal{O}_{D'}$ be an isomorphism of line bundles. Then each point admits an open neighbourhood U and a meromorphic function $f_U : U \to \bar{\mathbb{C}}$, such that $\mathcal{O}_D|U \to \mathcal{O}_{D'}|U$ is given by multiplication with f_U. The functions f_U glue to a meromorphic function f on X. We have $D' = D + (f)$. □

If D is a divisor on a compact Riemann surface than the degree

$$\deg(D) = \sum_{a \in X} D(a)$$

is defined. As we know, the degree of a principal divisor is zero. Hence $\deg D$ only depends on the divisor class. This enables us to define:

1.3 Remark. *Let $\mathcal{L}$ be a line bundle on a compact Riemann surface. Assume that there exists a divisor D such that $\mathcal{L} \cong \mathcal{O}_D$. The* ***degree*** Ldeg

$$\deg \mathcal{L} := \deg D$$

doesn't depend on the choice of D.

Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules. We denote by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ the set of all $\mathcal{O}_X$-linear maps $\mathcal{M} \to \mathcal{N}$. This is an $\mathcal{O}_X(X)$-module. More generally we can consider for every open $U \subset X$

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{M}|U, \mathcal{N}|U).$$

It is clear that this is presheaf. It is easy to check that it is actually a sheaf and moreover an $\mathcal{O}_X$-module. We denote it by

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}).$$

We denote by the $\mathcal{O}_X(U)^{p\times q}$ the set of all $p \times q$-matrices with entries from $\mathcal{O}_X(U)$. This is a free $\mathcal{O}_X$-module. There is an obvious natural isomorphism

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p,\mathcal{O}_X^q) \cong \mathcal{O}_X^{p\times q}.$$

Hence $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ is a vector bundle if $\mathcal{M}$ and $\mathcal{N}$ is a vector bundle. It is a line bundle if both are line bundles. The dual bundle of a vector-bundle $\mathcal{M}$ is

$$\mathcal{M}^* := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X).$$

It has the same rank as $\mathcal{M}$.

There is another construction which rests on the tensor-product of modules. Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules. The assignment

$$U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

defines a presheaf. This is usually not a sheaf. Hence we consider the generated sheaf and call it by $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$. Clearly this is an $\mathcal{O}_X$-module. The notion of an $\mathcal{O}_X$-bilinear map $\mathcal{M} \times \mathcal{N} \to \mathcal{P}$ for $\mathcal{O}_X$-modules $\mathcal{M},\mathcal{N},\mathcal{P}$ and the following universal property should be clear: For an $\mathcal{O}_X$-bilinear map $\mathcal{M} \times \mathcal{N} \to \mathcal{P}$ of $\mathcal{O}_X$-modules there exist a unique commutative diagram

$$\begin{array}{ccc} \mathcal{M}\times\mathcal{N} & \longrightarrow & \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N} \\ & \searrow \quad \swarrow & \\ & \mathcal{P} & \end{array}$$

with an $\mathcal{O}_X$-linear map $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{P}$.

One also has $\mathcal{O}_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_X^m \cong \mathcal{O}_X^{n\times m}$. Since the construction of the tensor product is compatible with the restriction to open subsets,

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|U \cong \mathcal{M}|U \otimes_{\mathcal{O}_X|U} \mathcal{N}|U,$$

we obtain that the tensor product of two vector bundles is a vector bundle and moreover the tensor product of two line bundles is a line bundle.

When M is an R-module and $M^* = \mathrm{Hom}_R(M,R)$ its dual, one has a natural bilinear map $M \times M^* \longrightarrow R$, $(a,l) \mapsto l(a)$. This induces a linear map $M \otimes_R M^* \to R$. Sheafifying we get an $\mathcal{O}_X$-bilinear map

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^* \longrightarrow \mathcal{O}_X.$$

Clearly this is an isomorphism when $\mathcal{M}$ is a line-bundle.

We denote by $[\mathcal{L}]$ the class of all line bundles which are isomorphic to $\mathcal{L}$. We denote be $\operatorname{Pic}(X)$ (Picard group) the set of all isomorphy classes of line-bundles. Since constructions as the "set of sets" are forbidden in set theory, there seems to be a logical difficulty. But this is not really there because one can easily prove that there exists a *set* of line-bundles such that every line bundle is isomorphic to one of this set.

We define the tensor product in $\operatorname{Pic}(X)$ by

$$[\mathcal{L}] \otimes [\mathcal{L}'] := [\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'].$$

This is clearly well-defined. We notice that this product is associative and commutative.

The element $[\mathcal{O}_X]$ is neutral element in $\operatorname{Pic}(X)$. Finally we notice that $[\mathcal{L}] \otimes [\mathcal{L}^*] = [\mathcal{O}_X]$. Hence every element of $\operatorname{Pic}(X)$ has an inverse.

1.4 Remark. *The set* $\operatorname{Pic}(X)$ *of isomorphy classes of line bundles is an abelian group under the tensor product.* Ilig

2. The finiteness theorem

We introduce the notion of a coherent sheaf.

2.1 Definition. *An* $\mathcal{O}_X$*-module* $\mathcal{M}$ *on a Riemann surface* X *is called a skyscraper sheaf if* $\mathcal{M}_a$ *is zero for all* a *outside a discrete set and if* $\mathcal{M}_a$ *is a finite dimensional complex vector space for all* a. DSs

The skyscraper sheaf is determined by the stalks:

2.2 Remark. *For a skyscraper sheaf the natural map* Skl

$$\mathcal{M}(X) \xrightarrow{\sim} \prod_{a \in X} \mathcal{M}_a$$

is an isomorphism.

Proof. The map is injective because $\mathcal{M}$ is a sheaf. To prove the surjectivity, we consider the discrete set S of all a with $\mathcal{M}_a \neq 0$. Let $(s_a) \in \prod \mathcal{M}_a$. We choose for each $a \in X$ an open neighborhood $U(a)$ such that $U(a) \cap S$ is $\{a\}$ or empty and such that s_a can be represented by a section $s_{U(a)} \in \mathcal{M}(U(a))$. This sections and the zero section on $X - S$ glue to a global section. □

The notion of a "coherent sheaf" includes skyscraper sheafs and vector bundles.

2.3 Definition. *A **coherent sheaf** on a Riemann surface is an $\mathcal{O}_X$-module such that there exists an exact sequence* DCs

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

with a skyscraper sheaf $\mathcal{W}$ and a vector bundle $\mathcal{N}$.

For a better understanding it is useful to look at the stalks:

$$0 \longrightarrow \mathcal{W}_a \longrightarrow \mathcal{M}_a \longrightarrow \mathcal{N}_a \longrightarrow 0.$$

Since $\mathcal{W}_a$ and $\mathcal{N}_a$ are finitely generated modules over $R = \mathcal{O}_{X,a}$, the module $\mathcal{M}_a$ is also finitely generated. We recall that every finitely generated R-module is isomorph to a finite product of modules $R/\mathfrak{a}$. When $\mathfrak{a}$ is different form 0 this is a finite dimensional vector space. An element a of an R-module is called *torsion element*, if there exists $r \in R$, $r \neq 0$ such that $ra = 0$. The set of all torsion elements is the torsion submodule M^{tor}. If M is finitely generated then M is isomorphic to the product of M^{tor} and a free module R^n. The number n is uniquely determined and is called the rank of M. An R-module M, which is an finite dimensional $\mathbb{C}$-vector space is a torsion module. Hence in 2.3 $\mathcal{W}_a$ must be a sub-module of $\mathcal{M}_a^{\mathrm{tor}}$. We claim that both are equal: For this we observe that the natural homomorphism

$$\mathcal{M}_a/\mathcal{W}_a \longrightarrow \mathcal{M}_a/\mathcal{M}_a^{\mathrm{tor}}$$

is a surjective module of free modules of the same rank. It is easy to see that such a homomorphism is an isomorphism. (The kernel is a free sub-module, its rank must be 0, hence it is 0.) We see that $\mathcal{W}$ in 2.3 is unique, more precisely

$$\mathcal{W}(U) = \left\{ s \in \mathcal{M}(U); \quad s_a \in \mathcal{M}_a^{\mathrm{tor}} \quad \text{for } a \in U \right\}.$$

This implies that "coherence" is a local property:

Let (U_i) be an open covering. An $\mathcal{O}_X$ module $\mathcal{M}$ is coherent if and only if all $\mathcal{M}|U_i$ are coherent.

2.4 Theorem. *Let $\mathcal{M}$ be a coherent sheaf on a compact Riemann surface. Then:* TE

a) $H^n(X, \mathcal{M})$ *is finite dimensional for all n.*
b) $H^n(X, \mathcal{M}) = 0$ *for $n > 1$.*

Proof. Skyscraper sheaves are flabby. The statement is clear in this case. Hence we can restrict to vector bundles:

Firstly we show the vanishing statement b). We will use the Dolbeault complex $0 \to \mathcal{O}_X \to A_X^0 \to A_X^{0,1} \to 0$. The essential point is that $\bar\partial : A_X^0 \to A_X^{0,1}$ is $\mathcal{O}_X$-linear. This comes from the fact that $\bar\partial f = 0$ for holomorphic functions. Hence we can consider the Dolbeault complex as sequence of $\mathcal{O}_X$-modules and we can tensor it with $\mathcal{M}$. The sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow A_X^0 \otimes_{\mathcal{O}_X} \mathcal{M} \to A^{0,1} \otimes_{\mathcal{O}_X} \mathcal{M} \to 0$$

remains of course exact. Since $A_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$ carries a natural structure as $\mathcal{C}_X^\infty$-module we have an acyclic resolution of $\mathcal{M}$.

Next we prove the finiteness of $H^0(X, \mathcal{M})$. We choose a finite open covering $\mathfrak{U} = (U_i)$ such that each U_i is biholomorphic equivalent to a disc in the plane. This is possible since X is compact. We can take the U_i so small that $\mathcal{M}|U_i$ is free. We choose a trivialization $\mathcal{M}|U_i \cong \mathcal{O}_U^n$. We also can assume that there is a relatively compact open subset $V_i \subset\subset U_i$ such that V_i is also biholomorphic to a disc and such that $\mathcal{V} = (V_i)$ still is a covering of X. We know that $\mathcal{M}(U_i)$ is a Frèchet space. A finite cartesian product of Frèchet spaces carries an obvious structure of a Frèchet space. Hence $\prod_i \mathcal{M}(U_i)$ is a Frèchet space. We can consider $\mathcal{M}(X)$ as a (closed) subspace of this space. Hence $\mathcal{M}(X)$ is a Frèchet space too. We can repeat this with the covering $\mathfrak{V}$. It is easy to see that both Frèchet structures on $\mathcal{M}(X)$ are the same. From Montel's theorem follows that the identity map $\mathcal{M}(X) \to \mathcal{M}(X)$ is compact. From I.7.6 follows that $\mathcal{M}(X)$ is finite dimensional.

Next we prove the finiteness of $H^1(X, \mathcal{M})$. From Leray's lemma we know

$$H^1(X, \mathcal{M}) = \check{H}^1(\mathfrak{U}, \mathcal{M}) = \check{H}^1(\mathfrak{V}, \mathcal{M}).$$

We consider the space of Čech cocycles $C^1(\mathfrak{U}, \mathcal{M})$. This is a closed subspace of $\prod_{ij} \mathcal{M}(U_i \cap U_j)$ and hence a Frèchet space. The same is true for $C^1(\mathfrak{V}, \mathcal{M})$. The natural restriction operator

$$r : C^1(\mathfrak{U}, \mathcal{M}) \longrightarrow C^1(\mathfrak{V}, \mathcal{M})$$

is a compact operator by Montel's theorem I.7.4. Now we consider the map

$$\begin{aligned} C^1(\mathfrak{U}, \mathcal{M}) \times \prod_i \mathcal{M}(V_i) &\longrightarrow C^1(\mathfrak{V}, \mathcal{M}), \\ (A, (s_i)) &\longmapsto r(A) + \delta((s_i)). \end{aligned}$$

This map is a continuous surjective linear map of Frèchet spaces. It differs from the map

$$\begin{aligned} C^1(\mathfrak{U}, \mathcal{M}) \times \prod_i \mathcal{M}(V_i) &\longrightarrow C^1(\mathfrak{V}, \mathcal{M}), \\ (A, (s_i)) &\longmapsto \delta((s_i)). \end{aligned}$$

only by a compact operator (essentially r). By the Schwartz theorem I.7.5 this map has finite dimensional cokernel. But the cokernel is $H^1(\mathfrak{V}, \mathcal{M})$. □

3. The Picard group

In section 1 we introduced the Picard group $\mathrm{Pic}(X)$, i.e. the group of isomorphism classes of line bundles. We want to compare it with the group $\mathrm{Div}(X)/H(X)$ of divisor classes. Recall that we associated to a divisor D on a Riemann surface X a line bundle $\mathcal{O}_D$. For two divisors D, D' there is a natural $\mathcal{O}_X$-bilinear multiplication map

$$\mathcal{O}_D \times \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{D+D'}.$$

It is clear that the induced homomorphism

$$\mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{D+D'}$$

is an isomorphism. Hence we get a *homomorphism* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. Since principal divisors are in the kernel (1.2) we get a homomorphism of the group of divisor classes into $\mathrm{Pic}(X)$. This homomorphism is injective (1.2). We claim more:

3.1 Proposition. *Let X be a compact Riemann surface. The natural homomorphism* DPiI

$$\mathrm{Div}(X)/H(X) \xrightarrow{\sim} \mathrm{Pic}(X)$$

is an isomorphism.

The proof is non-trivial and will use the finiteness theorem. We will use twists of sheaves. The twist of a coherent sheaf $\mathcal{M}$ with a line bundle $\mathcal{M}$ simply is $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$. We will use a very special twist. For the rest of the section we fix a point $a \in X$. For an integer n we consider the divisor (a)

$$(a)(x) = \begin{cases} 1 & \text{for } x = a, \\ 0 & \text{elswhere.} \end{cases}$$

We denote by $\mathcal{O}(n)$ the associated line bundle and by $\mathcal{M}(n)$ the twist of $\mathcal{M}$ with $\mathcal{O}(n)$.

Next we use the natural inclusion $\mathcal{O}(n) \hookrightarrow \mathcal{O}(n+1)$. It induces $\mathcal{M}(n) \hookrightarrow \mathcal{M}(n+1)$. We assume now for simplicity that $\mathcal{M}$ is a vector bundle. Then this map is injective and we have an exact sequence

$$0 \longrightarrow \mathcal{M}(n) \longrightarrow \mathcal{M}(n+1) \longrightarrow \mathcal{W}(n) \longrightarrow 0$$

with a skyscraper sheaf $\mathcal{W}(n)$. From the exact cohomology sequence we get that $H^1(X, \mathcal{M}(n)) \to H^1(X, \mathcal{M}(n+1))$ is surjective. By the finiteness theorem these are finite dimensional vector spaces. Hence we get:

3.2 Lemma. *Let $\mathcal{M}$ be a vector bundle on a compact Riemann surface X. For sufficiently large n the homomorphism* FslI

$$H^1(X,\mathcal{M}(n)) \to H^1(X,\mathcal{M}(n+1))$$

is an isomorphism

Now we make use of the exact sequence

$$H^0(X,\mathcal{M}(n+1)) \longrightarrow H^0(X,\mathcal{W}(n)) \longrightarrow H^1(X,\mathcal{M}(n)) \longrightarrow H^1(X,\mathcal{M}(n+1)).$$

For large n the last arrow is an isomorphism, hence $H^0(X,\mathcal{M}(n+1)) \longrightarrow H^0(X,\mathcal{W}(n))$ is surjective. Since $H^0(X,\mathcal{W}(n))$ is not zero we get:

3.3 Proposition. *Let $\mathcal{M}$ be a vector bundle of positive rank on a compact Riemann surface. Then $\mathcal{M}(n)$ admits for large enough n a non-zero global section.* LeGs

Taking products one gets a bilinear map $\mathcal{O}(-n)\times\mathcal{O}(n) \to \mathcal{O}_X$. More generally one gets for a vector bundle $\mathcal{M}$

$$\mathcal{M}\times\mathcal{O}(-n)\times\mathcal{M}^*\times\mathcal{O}(n) \longrightarrow \mathcal{O}_X.$$

This induces an $\mathcal{O}_X$-linear map

$$\mathcal{M}(-n)^* \xrightarrow{\sim} \mathcal{M}^*(n).$$

A local computation shows that this an isomorphism. For suitable (large enough) n one has a non-zero global section $S \in H^0(X,\mathcal{M}(n))$. We can use this section to define a non-zero $\mathcal{O}_X$-linear map

$$\mathcal{M}(-n) \longrightarrow \mathcal{O}_X, \quad (a \longmapsto S(a)).$$

Next we need:

3.4 Lemma. *Let $\mathcal{M}$ be a vector bundle on a Riemann surface and $\mathcal{M} \to \mathcal{O}_X$ a non-zero $\mathcal{O}_X$-linear map. Then the image is line bundle of the form $\mathcal{O}_{-D}$ with $D \geq 0$.* LbMd

We can replace X by a small open neighborhood of a given point a. Hence we may assume that $X = U$ is an open neighborhood of $a = 0$ in $\mathbb{C}$. Taking U small enough, we can assume $\mathcal{M} = \mathcal{O}_U^d$. The map $\mathcal{M} \to \mathcal{O}_U$ is given a system of holomorphic functions $f_1,\dots,f_d \in \mathcal{O}(U)$. We can assume that non of the f_i is zero and then that $f_i = z^{n_i} g_i$ where g_i is without zero. Modifying the map $\mathcal{M} \to \mathcal{O}_U$ we reduce to the case $g_i = 1$. Now we see that the image is $z^n\mathcal{O}_U$, where n is the minimum of the $n_1,\dots,n_d$. This proves 3.4. □

The same type of argument gives also information about the kernel. We can assume $n_1 \leq \dots \leq n_d$. But then the first component of a section of the kernel can be computed form the rest. Hence we get that the kernel again is free. This shows:

3.5 Coherence lemma. *let $\mathcal{M} \to \mathcal{L}$ an $\mathcal{O}_X$-linear map of a vector bundle into a line bundle (on a Riemann surface). Then the kernel is a vector bundle too.* CL

We go back to the map $\mathcal{M}(n) \to \mathcal{O}_X$. We know that the image is a line bundle. Tensoring with $\mathcal{O}(-n)$ we obtain:

3.6 Lemma. *Every vector bundle $\mathcal{M}$ of rank $n > 0$ sits in an exact sequence $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is a line bundle and $\mathcal{N}$ a vector bundle of rank $n-1$.* LVes

The proof shows a little more, namely that this line bundle can be taken in the form $\mathcal{O}_D$ for some divisor. Since a vector bundle of rank 0 is zero, we obtain the proof of 3.1. □

When we apply 3.3 to $\mathcal{M} = \mathcal{O}_X$ we get:

3.7 Theorem. *Every compact Riemann surface admits a non constant meromorphic function* EcRm

4. Riemann-Roch

without explicitly mentioning this. For a vector bundle on a Riemann surface we have defined the rank $n = \mathrm{Rank}(\mathcal{M})$. This can be generalized to a coherent sheaf $\mathcal{M}$: There exists a discrete subset $S \subset X$ such that $\mathcal{M}|(X-S)$ is a vector bundle. We define $\mathrm{Rank}(\mathcal{M}) := \mathrm{Rank}(\mathcal{M}|(X-S))$. The rank of a skyscraper sheaf is 0.

4.1 Remark. *For a short exact sequence of coherent sheaves* Drank

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

one has

$$\mathrm{Rank}(\mathcal{M}_2) = \mathrm{Rank}(\mathcal{M}_1) + \mathrm{Rank}(\mathcal{M}_3).$$

Proof. One has to use the following. Assume that R is an integral domain and

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow R^p \longrightarrow 0$$

an exact sequence of R-modules. Then $n = m + p$. This is well-known from linear algebra when R is a field and can be reduced to this case by imbedding R into a field. □

There are other quantities, which have the additive property as in 4.1. For a coherent sheaf we define

$$\chi(\mathcal{M}) = \sum_i (-1)^i \dim H^i(X,\mathcal{M}) = \dim H^0(X,\mathcal{M}) - \dim H^1(X,\mathcal{M}).$$

in analogy to 4.1 we have:

4.2 Remark. *For a short exact sequence of coherent sheaves* Drankc

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

one has

$$\chi(\mathcal{M}_2) = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_3).$$

Proof. One has to use the long exact cohomology sequence and the following simple result of linear algebra: Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow$$

en exact sequence of finite-dimensional vector spaces, then

$$\sum_{i=1}^n (-1)^i \dim V_i = 0. \qquad \square$$

4.3 Theorem. *let X be a compact Riemann surface. There exists a unique function which associates to an arbitrary coherent sheaf $\mathcal{M}$ on X a non-negative integer* $\deg \mathcal{M}$ *such that the following properties are satisfied:* Ddeg

1. $\deg \mathcal{M}$ *depends only on the isomorphy class of $\mathcal{M}$.*
2. *For a skyscraper sheaf $\mathcal{W}$*

$$\deg(\mathcal{W}) = \sum_{a\in X} \dim \mathcal{W}_a.$$

3. *If D is a divisor and $\mathcal{D}$ the associated line-bundle then*

$$\deg(\mathcal{O}_D) = \deg D.$$

4. *For a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

one has

$$\deg(\mathcal{M}_2) = \deg(\mathcal{M}_1) + \deg(\mathcal{M}_3).$$

We will prove this together with the

4.4 Riemann-Roch theorem. *Let $\mathcal{M}$ be a coherent sheaf on a compact Riemann surface X. Then* RR

$$\chi(\mathcal{M}) = \deg(\mathcal{M}) + \operatorname{Rank}(\mathcal{M})(1-g).$$

Corollary. $\dim H^0(X,\mathcal{M}) \geq \deg(\mathcal{M}) + \operatorname{Rank}(\mathcal{M})(1-g)$.

So far the degree has been defined for skyscraper sheaves. The Riemann-Roch theorem is trivial in the case. Since the rank of a skyscraper sheaf is 0 it reduces to the equation $\dim \mathcal{M}(X) = \deg \mathcal{M}$ which follows from 2.2.

The Riemann-Roch theorem is also trivial for $\mathcal{O}_X$, since

$$\chi(\mathcal{O}_X) = \dim H^0(X,\mathcal{O}_X) - \dim H^1(\mathcal{O}_X) = 1-g.$$

Next we prove the Riemann-Roch theorem for a divisor D. We add to the divisor D a point divisor (a). (This divisor is 1 in a and 0 elsewhere). We have an exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+(a)} \longrightarrow \mathcal{W} \longrightarrow 0$$

with a skyscraper sheaf $\mathcal{W}$. Since we have $\deg(D+(a)) = \deg(D)+\deg(\mathcal{W})$, the Riemann-Roch theorem holds for $\mathcal{O}_D$ if and only if it holds for $\mathcal{O}_{D+(a)}$. Adding and subtracting point divisors one can reduce the case of an arbitray D to the zero-divisor in a finite number of steps. But for the zero-divisor ($\mathcal{L} = \mathcal{O}_X$) Riemann-Roch has been proved.

We now make a little trick. We *define* $\deg(\mathcal{M})$ by the Riemann-Roch formula. For skyscraper sheaves and for line bundles this coincides with the definition we gave already. But also 4) in 4.3 is true since additivity holds for χ and Rank. The uniqueness of deg with the properties 1)-4) follows from 3.1 and 3.6. This proves the existence of the degree and the Riemann-Roch theorem.

We mention two other formulae for the degree.

4.5 Lemma. *Let $\mathcal{M},\mathcal{N}$ be a two vector bundles, then* Ffdd

$$\deg(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) = \deg \mathcal{M} \cdot \operatorname{Rank}\mathcal{N} + \deg \mathcal{N} \cdot \operatorname{Rank}\mathcal{M}.$$

Let $\mathcal{M}$, $\mathcal{N}$ be two vector bundles than

$$\deg(\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) = \deg \mathcal{N} - \deg \mathcal{M}.$$

Proof. For line bundles the statements are clear. The general case can be reduced to this one by induction on the ranks using exact sequences of the type as $0 \to \mathcal{M}_1 \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow 0$ where $\mathcal{L}$ is a line bundle (6.3.6). □

There is another useful formula. Since we will not use it, we keep short:

Let $\mathcal{M}$ be a vector bundle of rank n. One can define $\Lambda^p\mathcal{M}$ as a set of $\mathcal{O}_X$-multilinear alternating forms $\mathcal{M}^* \times \cdots \times \mathcal{M}^* \to \mathcal{O}_X$ (p-copies). It should be clear what this means. In the case $p=n$ one obtains a line bundle which sometimes is called the *determinant* of $\mathcal{M}$.

4.6 Remark. *A vector bundle and its determinant have the same degree.* Vdsd

We indicate the proof: If $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ is a short exact sequence then the determinant of $\mathcal{M}_2$ and $\mathcal{M}_1 \times \mathcal{M}_3$ are isomorphic. This allows to reduce the statement to the case where $\mathcal{M} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ is product of line bundles. But then the determinant equals $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$. □

5. A residue map

In this section X is a compact (and connected) Riemann surface. We now have to make use of the sheaf Ω_X of holomorphic differentials. For sake of simplicity we well write $\mathcal{O} = \mathcal{O}_X$, $\Omega = \Omega_X$. Basic will be the „residue map“

$$\operatorname{Res} : H^1(X, \Omega) \longrightarrow \mathbb{C}.$$

5.1 Remark and first definition of the. *Consider the exact sequence* FdR

$$0 \longrightarrow \Omega \longrightarrow A^{1,0} \xrightarrow{\partial} A^2 \longrightarrow 0$$

and the resulting isomorphism

$$H^1(X, \Omega) \cong A^2(X)/\partial A^{1,0}.$$

For an $\xi \in H^1(X, \Omega)$ consider a representant $\omega \in A^2(X)$. The integral

$$\operatorname{Res} \xi := \frac{1}{2\pi \mathrm{i}} \int_X \omega$$

is independent of the choice of the representant ω.

The independence follows from Stokes' theorem $\int_X d\omega = 0$. Notice that on the level of $A^{1,0}$ we have $\partial = d$.

We come to another property which justifies the notation “residue”. For this we consider the sheaf $\mathfrak{M}$ of meromorphic 1-forms and the exact sequence

$$0 \longrightarrow \Omega \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}/\Omega \longrightarrow 0.$$

This induces the map

$$H^0(X, \mathfrak{M}/\Omega) \xrightarrow{\delta} H^1(X, \Omega).$$

We define now a residue map

$$\operatorname{Res} : H^0(X, \mathfrak{M}/\Omega) \longrightarrow \mathbb{C}.$$

Recall that the elements of $(\mathfrak{M}/\Omega)(X)$ are families $(\omega_a)_{a \in X}$, $\omega \in \mathcal{M}_a/\mathcal{O}_a$, which locally fit together. This means that there exists an open covering $\mathfrak{U} = (U_i)$ and meromorphic differentials $\omega_i \in \mathfrak{M}(U_i)$, which represent the ω_a for all $a \in U_i$. The differences $\omega_i - \omega_j$ are holomorphic in $U_i \cap U_j$. Hence the residue $\operatorname{Res}(\omega_a, a)$ is well defined.

5.2 Remark and second definition of the. *There is a natural linear map* SdR

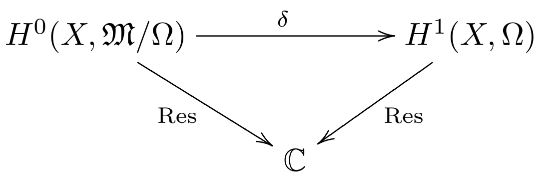
$$\operatorname{Res} : H^0(X, \mathfrak{M}/\Omega) \longrightarrow \mathbb{C},$$

which can be defined as

$$\operatorname{Res}((\omega_a)) = \sum_{a \in X} \operatorname{Res}(\omega_a, a).$$

Both definitions of the residue are related:

5.3 Proposition. *The diagram* Resz



commutes.

Proof. We start with an element from $H^0(X, \mathfrak{M}/\Omega)$. As we have seen there exists an open covering (U_i) and meromorphic differentials ω_i on U_i, such that $\omega_{ij} := \omega_i - \omega_j$ is holomorphic on $U_i \cap U_j$. The family ω_{ij} is a Čech cocycle with coefficients in Ω and defines an element of $H^1(X, \Omega)$. This is the image of $(\omega_i) \in H^0(X, \mathfrak{M}/\Omega)$ under the combining map $\delta : H^0(X, \mathfrak{M}/\Omega) \longrightarrow H^1(X, \Omega)$. We can consider the forms (ω_{ij}) also as a Čech-cocycle of $A^{1,0}$. Since the first cohomology of this sheaf vanishes there exist elements $\eta_i \in A^{1,0}(U_i)$ such that

$$\eta_i - \eta_j = \omega_{ij} \quad (= \omega_i - \omega_j)$$

The advantage of the η_i compared to the ω_i is that they have no poles in U_i. The prize which we have to pay is that they are not holomorphic. Since $\omega_i - \omega_j$ is holomorphic we have $d\omega_i = d\omega_j$ and hence $d\eta_i = d\eta_j$ on $U_i \cap U_j$. Hence the $d\eta_i$ glue to a differential $\omega \in A^2(X)$. This defines a cohomology class in $H^1(X, \Omega) = A^2(X)/\partial A^{1,0}$. One can check (we leave this to the reader) that this cohomology class corresponds to the cocycle (ω_{ij}).

We have to compute $\int_X \omega$ and to relate it to the residues of (ω_i). Hence we consider the finite set of poles S. The differences $\eta_i - \omega_i$ are smooth on $X' = X - S$ and glue to a differential $\eta \in A^{1,0}(X')$. We have $d\eta = -\omega$ on X'. Hence $d\eta$ (but not η) has a smooth extension to X.

We now consider the integral $\int_X \omega$. We cut out the disc of radius ε around each $a \in S$ (with respect to some chart) and denote the complement of the discs by $X(\varepsilon)$. We have

$$\int_X \omega = \lim_{\varepsilon \to 0} \int_{X(\varepsilon)} \omega.$$

Now Stokes' formula applies and gives

$$\int_{X(\varepsilon)} \omega = \sum_{a\in S} \oint_\alpha \eta,$$

where α denotes a small circle around a (mathematically negative orientation).

The proof of 5.3 will be complete if we show

$$\lim_{\varepsilon\to 0} \oint_\alpha \eta = 2\pi\mathrm{i}\,\mathrm{Res}\,\omega_i \qquad (a \in U_i).$$

Here we have chosen i such that $a \in U_i$. On a small punctured neighborhood of a we have $\eta = \eta_i - \omega_i$. Since the integral of $-\omega_i$ produces the residue, it remains to show

$$\lim_{\varepsilon\to 0} \oint_\alpha \eta_i = 0.$$

But this is clear since η_i is smooth on U_i (including a). □

5.4 Proposition. *The residue map $H^1(X,\Omega) \to \mathbb{C}$ is different from zero.* Rmdz

Proof. Because of 5.3 it is sufficient to show that the map $H^0(X,\mathfrak{M}/\Omega) \to \mathbb{C}$ is not zero. For this one takes a point $a \in X$ and a meromorphic differential ω_1 on a small open neighborhood U_1 which has a simple pole at a and no other pole. Then we consider the zero form ω_2 on $U_2 = X - \{a\}$. Both together glue to a global section of $\mathfrak{M}/\Omega$. Its residue is different from 0. □

6. Serre duality

Let $\mathcal{M}$ be a vector bundle on the compact Riemann surface X. We consider the vector bundle $\mathscr{H}om(\mathcal{M},\Omega)$. There is a natural $\mathcal{O}_X$-bilinear map

$$\mathcal{M} \times \mathscr{H}om(\mathcal{M},\Omega) \longrightarrow \Omega.$$

Locally it is given by $(m,l) \mapsto l(m)$. For a fixed global section from $\mathscr{H}om(\mathcal{M},\Omega)$ this induces a map

$$\mathcal{M} \longrightarrow \Omega.$$

Taking cohomology we get a map

$$H^1(X,\mathcal{M}) \longrightarrow H^1(X,\Omega).$$

Varying the global section we get a bilinear map

$$H^0(X, \mathscr{H}om(\mathcal{M},\Omega)) \times H^1(X,\mathcal{M}) \longrightarrow H^1(X,\Omega).$$

Combining it with the residue map we get

$$H^0(X, \mathscr{H}om(\mathcal{M}, \Omega)) \times H^1(X, \mathcal{M}) \longrightarrow \mathbb{C}.$$

This can be considered as a linear map

$$H^0(X, \mathscr{H}om(\mathcal{M}, \Omega)) \longrightarrow H^1(X, \mathcal{M})^*,$$

where V^* denotes the dual of a vector space ($V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$). We call this map the *duality map.*

6.1 Serre duality. *For a vector bundle $\mathcal{M}$ on a compact Riemann surface the duality map* SD

$$H^0(X, \mathscr{H}om(\mathcal{M}, \Omega)) \longrightarrow H^1(X, \mathcal{M})^*$$

is an isomorphism.

Corollary. $\dim H^1(X, \mathcal{M}) = \dim H^0(X, \mathscr{H}om(\mathcal{M}, \Omega))$.

Proof. In a first step we prove the injectivity of the duality map. Let $f : \mathcal{M} \longrightarrow \Omega$ be a non-zero $\mathcal{O}$-linear map. We use the sequence

$$0 \longrightarrow \mathrm{Kernel}(f) \longrightarrow \mathcal{M} \longrightarrow f(\mathcal{M}) \longrightarrow 0.$$

We know that $f(\mathcal{M})$ and Kernel(f) are vector bundles, 3.4 and 3.5. Since f is non-trivial, $f(\mathcal{M})$ is a line bundle. Since $H^2(X, \mathrm{Kernel}(f)) = 0$ we get, that $H^1(X, \mathcal{M}) \to H^1(X, f(\mathcal{M}))$ is surjective. Now we use the sequence

$$0 \longrightarrow f(\mathcal{M}) \longrightarrow \Omega \longrightarrow f(\mathcal{M})/\Omega \longrightarrow 0.$$

Since $f(\mathcal{M})/\Omega$ is a skyscraper sheaf ,its first cohomology group vanishes and we get the surjectivity of $H^1(X, f(\mathcal{M})) \to H^1(X, \Omega)$. As a consequence the composition $H^1(\mathcal{M}) \longrightarrow H^1(\Omega)$ is surjective. Because of 5.4 the composition with the residue map $H^1(\mathcal{M}) \to \mathbb{C}$ is non-zero. (It is easy to check that this is the image of f under the duality map.) □

Next we will proof the surjectivity of the duality map. This more involved, since it needs some control that $H^1(X, \mathcal{M})$ is not to big. The idea is to consider twists $\mathcal{M}(-n)$. Recall that to define $\mathcal{M}(-n)$ one has to choose a point a and then to consider the divisor $-n(a)$ which is concentrated on a with multiplicity $-n$. The associated line bundle $\mathcal{O}(-n)$ is contained in $\mathcal{O}$, when $n \geq 0$. Hence $\mathcal{M}(-n)$ can be considered as subsheaf of $\mathcal{M}$ for $n \geq 0$. Especially $\dim H^0(X, \mathcal{M}(-n)) \leq \dim H^0(X, \mathcal{M})$. Now we apply Riemann Roch to $\mathcal{M}(-n)$. We use the formula $\deg(\mathcal{M}(-n)) = \deg(\mathcal{M}) - n \cdot \mathrm{Rank}\mathcal{M}$ (4.5). Of course $\mathrm{Rank}(\mathcal{M}) = \mathrm{Rank}(\mathcal{M}(n))$. Hence Riemann-Roch gives

$$\begin{aligned}\dim H^1(X, \mathcal{M}(-n)) = &\dim H^0(X, \mathcal{M}(-n)) + n \cdot \mathrm{Rank}\mathcal{M} \\ &- \deg(\mathcal{M}) + \mathrm{Rank}(\mathcal{M})(1 - g).\end{aligned}$$

As we have seen, $\dim H^0(X, \mathcal{M}(-n))$ is bounded for $n > 0$.

6.2 Lemma. *For $n \to \infty$ the asymptotic formula* Afh

$$\dim H^1(X, \mathcal{M}(-n)) = n \cdot \mathrm{Rank}\mathcal{M} + O(1)$$

holds.

Similarly we get the asymptotic behavior of $\dim H^0(X, \mathscr{H}om(\mathcal{M}(-n), \Omega))$ by means of Riemann Roch. One has $\deg \mathscr{H}om(\mathcal{M}(-n), \Omega) = n\mathrm{Rank}\mathcal{M} - \deg \mathcal{M} + \deg(\Omega)$. The inclusion $\mathcal{M}(-n) \hookrightarrow \mathcal{M}$ gives a surjection $\mathscr{H}om(\mathcal{M}, \Omega) \to \mathscr{H}om(\mathcal{M}(-n), \Omega)$ and this induces a surjection in the first cohomology (vanishing of H^2). We get the boundedness of $\dim H^1(X, \mathcal{M}(n))$. Now Riemann-Roch shows the following asymptotic formula, which can be considered as a weak form of the duality theorem (together with 6.2).

6.3 Lemma. *For $n \to \infty$ the asymptotic formula* Afl

$$\dim H^0(X, \mathscr{H}om(\mathcal{M}(-n), \Omega)) = n \cdot \mathrm{Rank}\mathcal{M} + O(1)$$

holds.

Another asymptotic formula which immediately follows from Riemann Roch is

6.4 Lemma. *For $n \to \infty$ the asymptotic formula* Af10

$$\dim H^0(X, \mathcal{O}(n)) = n + O(1)$$

holds.

Next we have to investigate the behavior of the duality map under twisting. From the exact sequence $0 \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{M}/\mathcal{M}(-1) \to 0$ follows

$$H^0(X, \mathcal{M}/\mathcal{M}(-1)) \longrightarrow H^1(X, \mathcal{M}(-1)) \longrightarrow H^1(X, \mathcal{M}(-1))$$

Dualizing gives the exact sequence

$$H^1(X, \mathcal{M}(-1))^* \longrightarrow H^1(X, \mathcal{M}(-1))^* \longrightarrow (\mathcal{M}_a/\mathcal{M}(-1)_a)^*.$$

On the other side the imbedding $\mathcal{M}(-1) \subset \mathcal{M}$ gives a map

$$\mathscr{H}om_{\mathcal{O}}(\mathcal{M}, \Omega) \longrightarrow \mathscr{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega).$$

Next we use the fact that an $\mathcal{O}$-linear map of $\mathcal{O}$-modules $\mathcal{M}_1 \to \mathcal{M}_2$ induces a natural map (actually an isomorphism) $\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{L} \to \mathcal{M}_2 \otimes_{\mathcal{O}} \mathcal{L}$. Hence we can identify

$$\mathscr{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega) = \mathscr{H}om_{\mathcal{O}}(\mathcal{M}, \Omega(1)).$$

Let Ω' be the sheaf of meromorphic differentials which are holomorphic outside a and have possibly in a a pole of order at most one. There is an obvious isomorphism $\mathcal{M}(1) \longrightarrow \mathcal{M}'$, which sends $\omega \otimes f$ to $f\omega$. Let $\varphi : \mathcal{M} \to \Omega(1)$ an $\mathcal{O}$-linear map. It induces a map $\mathcal{M}_a \to \Omega(1)_a$. We can combine it with the residue map $\Omega(1)_a \to \mathbb{C}$. The map $\varphi : \mathcal{M}_a \to \mathbb{C}$ is zero on $\mathcal{M}_a(-1)$. Hence it induces a map $\varphi : \mathcal{M}_a/\mathcal{M}(-1)_a \to \mathbb{C}$. This gives is a map

$$H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M}, \Omega(1))) \longrightarrow (\mathcal{M}_a/\mathcal{M}(-1)_a)^*.$$

6.5 Lemma. *The diagram* Mtec

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(X,\mathcal{M})^* & \longrightarrow & H^1(X,\mathcal{M}(-1))^* & \longrightarrow & (\mathcal{M}_a/\mathcal{M}(-1)_a)^* \\ & \uparrow & & \uparrow & & \| \uparrow \\ 0 \longrightarrow & H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M},\Omega)) & \longrightarrow & H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M}(-1),\Omega)) & \longrightarrow & (\mathcal{M}_a/\mathcal{M}(-1)_a)^* \\ & \uparrow & & \uparrow & & \\ & 0 & & 0 & & \end{array}$$

is commutative. Its lines and columns are exact.

Let $\lambda \in H^1(X,\mathcal{M})^*$. Assume that the image in $H^1(X,\mathcal{M}(1))^*$ comes from $H^0(\mathscr{H}om_{\mathcal{O}}(\mathcal{M}(-1),\Omega))$. Simple diagram chasing in 6.5 shows that then λ comes from $H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M},\Omega))$. We can apply this several times and obtain the following

6.6 Lemma. *Let $\lambda \in H^1(X,\mathcal{M})^*$ and $D \geq 0$ a divisor. We write $\mathcal{M}_{-D} :=$* Lkeh
$\mathcal{M} \otimes \mathcal{O}_{-D}$. (This is imbedded in $\mathcal{M}$.) Assume that the image of λ under $H^1(X,\mathcal{M})^* \hookrightarrow \lambda \in H^1(X,\mathcal{M}_{-D})^*$ comes from $H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M}_{-D},\Omega))$. Then λ comes from $H^0(X, \mathscr{H}om_{\mathcal{O}}(\mathcal{M},\Omega))$.

We need a last preparation of the proof of the duality theorem. Assume that a function $f \in H^0(X,\mathcal{O}(n))$ is given, i.e. a meromorphic function which is holomorphic outside a ans has in a at most a pole of order $n > 0$. Multiplication by f defines a $\mathcal{O}$-linear map $f : \mathcal{M}(-n) \to \mathcal{M}$. (We considered up to now this map for $f = 1$.) This map induces a linear map

$$f : H^1(X,\mathcal{M})^* \longrightarrow H^1(X,\mathcal{M}(-n))^*.$$

Now we fix a non-zero element $\lambda \in H^1(X,\mathcal{M})^*$. (Duality states that this is in the image of the duality map.) We use it to define a map $H^0(X,\mathcal{O}(n)) \longrightarrow H^1(X,\mathcal{M}(-n))^*$, $f \longmapsto f(\lambda)$.

6.7 Lemma. *The map* Lmii

$$H^0(X,\mathcal{O}(n)) \longrightarrow H^1(X,\mathcal{M}(-n))^*, \quad f \longmapsto f(\lambda),$$

is injective.

Proof. We have to show that $f(\lambda) \neq 0$ for $f \neq 0$. This can also be interpreted as follows. For fixed $f \neq 0$ the map $f : H^1(X, \mathcal{M})^* \longrightarrow H^1(X, \mathcal{M}(-n))^*$ is injective, or equivalently, the map $f : H^1(X, \mathcal{M}(-n)) \longrightarrow H^1(X, \mathcal{M})$ is surjective. This follows from the long exact cohomology sequence since $\mathcal{M}/f(\mathcal{M}(-n)$ is a skyscraper sheaf. □

Now we consider the triangle

$$\begin{array}{ccc} H^0(X, \mathcal{O}(n)) & \xrightarrow{f \mapsto f(\lambda)} & H^1(X, \mathcal{M}(-n))^* \\ & & \uparrow \\ & & H^0(\mathcal{H}om(\mathcal{M}(-n)), \Omega) \end{array}$$

Now we use the asymptotic formulas 6.2, 6.3 and 6.4. They show that the images of $H^0(X, \mathcal{O}(n))$ and $H^0(\mathcal{H}om(\mathcal{M}(-n)), \Omega)$ for large n have a non-zero intersection in $H^1(X, \mathcal{M}(-n))$. Therefore for large n there must exist a non-zero $f \in H^0(X, \mathcal{O}(n))$ such that $f(\lambda)$ is in the image of the vertical arrow (=duality map for $\mathcal{M}(n)$). We consider the divisor $D = n(a) + (f)$ with the property $D \geq 0$. Hence $\mathcal{M}_{-D}$ is naturally imbedded in $\mathcal{M}$. Multiplication with f gives already a map $f : \mathcal{M}_{-D} \to \mathcal{M}$. Hence the map $f : H^1(X, \mathcal{M})^* \to H^1(X, \mathcal{M}(-n))^*$ factors as

$$\begin{array}{ccccc} H^1(X, \mathcal{M})^* & \overset{\text{natural imbedding}}{\longrightarrow} & H^1(X, \mathcal{M}_{-D})^* & \overset{f}{\longrightarrow} & H^1(X, \mathcal{M}(-n))^* \\ \lambda & \longmapsto & \lambda & \longmapsto & f(\lambda) \end{array}$$

In the same way we get a factorization

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \overset{\text{nat. i.}}{\longrightarrow} H^0(X, \mathcal{H}om(\mathcal{M}_{-D}, \Omega)) \overset{f}{\longrightarrow} H^0(X\, \mathcal{H}om(\mathcal{M}(-n), \Omega)).$$

This shows that $\lambda \in \mathcal{H}om(\mathcal{M}_{-D}, \Omega)$ is contained in the image of the duality map. Now 6.6 applies and shows that already $\lambda \in \mathcal{H}om(\mathcal{M}, \Omega)$ is contained in the duality map. This completes the proof of the duality theorem. □

7. Some comments on the Riemann-Roch theorem

The Riemann-Roch theorem can now be written in the cohomology free form

$$\boxed{\dim H^1(X, \mathcal{M}) - \dim H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) = \deg(\mathcal{M}) + \operatorname{Rank}(\mathcal{M})(1 - g).}$$

We formulate it in its classical form for divisors. A divisor K is called *canonical* if the associated line bundle is isomorphic to Ω. One can get a canonical divisor as follows: Let ω be a meromorphic 1-form. It is clear how to associate to ω a divisor $D = (\omega)$ which describes the poles and zeros of ω. This is obviously a canonical divisor, since the sections of $\mathcal{O}_D$ are the meromorphic functions f with the property that $f\omega$ is holomorphic. This gives an isomorphism $\mathcal{O}_D \cong \Omega$. By the way, a non vanishing meromorphic 1-form exists. One can take one of the form df, where f is a non-constant meromorphic function, which exists by 3.7.

Using the notation

$$l(D) := H^0(X, \mathcal{O}_D) = \dim\{\, f \text{ meromorphic on } X; \quad (f) \ge -D \,\}$$

we now get

7.1 Classical Riemann-Roch. *Let D be a divisor on a compact Riemann surface and let be K a canonical divisor. Then* CRR

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

We consider some special cases: let D be the zero divisor. Then $l(D) = 1$, $\deg(D) = 0$ and we obtain a different definition for the genus.

7.2 Theorem. *One has* $\dim \Omega(X) = g$. dOg

We now take $D = K$ in the Riemann-Roch formula. We get $g - 1 = \deg(D) + 1 - g$:

7.3 Theorem. *The degree of a canonical divisor (or of Ω) is $2g - 2$.* dCD

If f is a non zero meromorphic function with the property $(f) + D \ge 0$, then $\deg(D) \ge 0$. Hence $l(D) = 0$ for $\deg(D) < 0$. Applying this to $K - D$ we get:

7.4 Theorem. *Let D be a divisor with* $\deg(D) > 2g - 2$. *Then* RRv

$$l(D) = \deg(D) + 1 - g.$$

Finally we derive a formula which allows to compute the genus: Assume that $f : X \to Y$ is a non-constant holomorphic map of compact Riemann surfaces. Let $a \in X$. Recall that one can define the multiplicity $v(f, a)$ of f in a. We call

$$b(f, a) := v(f, a) - 1$$

the ramification order. This is zero if and only if f is locally biholmorphic at a. We denote by

$$b(f) = \sum_{a \in X} b(f, a)$$

the *total ramification order.* The sum is of course finite.

7.5 Hurwitz ramification formula. *Let $f : X \to Y$ a non-constant holomorphic map of compact Riemann surfaces. Let n be the covering degree of f (s. V.4.7). Then the following relation between the genus g_X of X and g_Y of Y holds:* HVf

$$\boxed{g_X = \tfrac{b(f)}{2} + n(g_Y - 1) + 1.}$$

We just give a short hint for the proof. Consider a meromorphic function g on Y and compare the divisors of the differentials dg on Y and $d(g \circ f)$ on X. Use 7.3. □

We treat some simple examples.

1) Since $H^1(\bar{\mathbb{C}}, \mathcal{O}) = 0$ (IV.4.10) we have that the genus of the Riemann sphere is zero.

2) A meromorphic differential on $\bar{\mathbb{C}}$ is given by dz. It has pole of order 2 in ∞ (use the chart $1/z$ and the fact that the derivative of $1/z$ is $-1/z^2$) and no further poles or zeros. Hence the degree of the canonical divisor is -2 in concordance with the expression $2g - 2$.

3) Consider the map $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $z \mapsto z^2$. There are two ramification points $0, \infty$ of ramification order 1. The covering degree is 2, the total ramification number is also 2. The Hurwitz formula gives $0 = 1 + 2(0 - 1) + 1$ which is correct.

4) Consider a complex torus $X = \mathbb{C}/L$. The differential dz on $\mathbb{C}$ is invariant under translations. Hence it gives a holomorphic differential on X. It has no poles and zeros, hence its degree is zero. From $2g - 2 = 0$ we get $g = 1$. Hence the genus of a torus is one.

5) Another way to see it is to use the Weierstrass $\wp$-function. This is holomorphic map $\mathbb{C}/L \to \bar{\mathbb{C}}$ with 4 ramification points (the zeros of $\wp'$ and ∞). The covering degree is 2, the total ramification number is 4. The Hurwitz formula gives $g = 2 + 2(-1) + 1 = 1$.

5) At the end of chapter V and section 5 we explained the Riemann surface of the “function $\sqrt{P(z)}$”, where P is polynomial without multiple zeros. Let the degree of P be $n = 2g+1$ or $n = 2g+2$. We explained that this Riemann surface is a two fold covering $f : X \to \bar{\mathbb{C}}$. The number b of ramification points (all of order two) turned out to be $n \le b \le n + 1$. Since we know from the Hurwitz formula 6.7.5 that the total ramification order is even, we

can conclude that in both cases $b = 2g$ is the correct number. From the Hurwitz formula we get:

The genus of the Riemann surface "$\sqrt{P(z)}$" is g, where where the degree of P is $2g+1$ or $2g+2$.

We consider the projection $z : X \to \bar{\mathbb{C}}$. Then dz is a meromorphic differential on X. Recall that $\sqrt{P(z)}$ has been defined as a meromorphic function on X. Hence we can consider the meromorphic differentials on X

$$\frac{z^k dz}{\sqrt{P(z)}}.$$

If one goes carefully through the construction of X one can work out when this differential has no poles. The result is that this is the case if and only if

$$0 \le k < g.$$

Since these differentials are linearly independent, by 7.2 they must give a basis of $\Omega(X)$.

The integration of these integrals is a major problem. In the case $g = 0$ one has so-called circle integrals. The integration is elementary and leads to the functions sin and cos. The case $g = 1$ is more involved. The integrals are called elliptic integrals. Their integration leads to the theory of elliptic functions. In the language of Riemann surfaces this means that one can show that X is biholomorhic equivalent to a complex torus $\mathbb{C}/L$ and that every complex torus arises in this way. (The fact that both types have genus 1 may be considered as a weak hint that this is true.) In the case $g > 1$ the integrals are called "hyperelliptic integral" and the corresponding Riemann surfaces are called *hyperelliptic Riemann surfaces*. The integration of hyperelliptic differentials leads deep into the theory of Riemann surfaces. With their help deep open problems as the Jacobi inversion problem could be solved.

Chapter VII. The Jacobi inversion problem

1. Harmonic differentials

Recall that a function u on an open subset $U \subset \mathbb{C}$ is called *harmonic,* if it satisfies

$$\Delta u := \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) u = 0.$$

A function is harmonic if and only if its real and imaginary part are harmonic. It is known from complex calculus that a real function u is harmonic if and only if it is locally the real part of an analytic function. This enables a quick generalization to Riemann surfaces.

1.1 Definition. *A real function f on a Riemann surface X is called harmonic, if every point admits an open neighborhood U such that u can be written an U as real part of an analytic function $f_U : U \to \mathbb{C}$. An arbitray (complex valued function u is called harmonic if its real and imaginary part are harmonic.* DhaR

We also need the notion of a harmonic differential.

1.2 Definition. *A differential $\omega \in A^1(X)$ on a Riemann surface is called harmonic, if every point admits an open neighborhood U such that ω can be written on U as du with a harmonic function $u : U \to \mathbb{C}$.* DhDaR

Analytic functions are also harmonic. Hence we obtain that holomorphic differentials are also harmonic.

We define the complex conjugate $\bar\omega$ of a differential ω. In coordinates this is defined as

$$\overline{f dx + g dy} := \bar f dx + \bar g dy.$$

One checks $\gamma^*\bar\omega = \overline{\gamma^*\omega}$ for arbitrary differential maps. This allows to generalize complex conjugation on Riemann surfaces via charts. A differential ω is called *real* if $\omega = \bar\omega$. The formula $\overline{dz} = d\bar z$ shows that complex conjugation defines an isomorphism

$$A^{1,0}(X) \xrightarrow{\sim} A^{0,1}(X), \qquad \omega \longmapsto \bar\omega.$$

The complex conjugate of a harmonic differential is harmonic as well. Hence we see that not only holomorphic differentials are harmonic bur also antiholomorphic ones. (A function or a differential is called antiholomorphic if its complex conjugate is holomorphic.)

1.3 Remark. *A harmonic differential ω can be written as the sum of a holomorphic and a antiholomorphic differential.* Hsha

Proof. Firstly we notice that a harmonic function locally is the sum of a holomorphic and a antiholomorphic function. This shows that a harmonic differential locally can be written in the form $\omega = \omega_1 + \omega_2$ with a holomorphic differential ω and an antiholmorphic differential ω_2. Since this decomposition is obviously unique i extends to a global decomposition. □

We introduce the so-called star operator. Recall that we have the direct decomposition

$$A^1(X) = A^{1,0}(X) \oplus A^{0,1}(X).$$

1.4 Definition. *The **star operator*** DstO

$$* : A^1(X) \longrightarrow A^1(X)$$

is defined by

$$*(\omega_1 + \omega_2) = \mathrm{i}(\bar{\omega}_1 - \bar{\omega}_2), \qquad \omega_1 \in A^{1,0}(X), \quad \omega_2 \in A^{0,1}(X).$$

Before we continue we formulate some rules which we will use in the following:

1.5 Some rules. SR

a) $d(f\omega) = (df) \wedge \omega + f\omega \qquad (f \in \mathcal{C}^\infty(X),\ \omega \in A^1(X))$.

b) *The same as in* a) *but d replaced by ∂ or $\bar\partial$.*

c) $* * \omega = -\omega$.

d) $\overline{*\omega} = *\bar{\omega}$.

e) $d(*\omega) = \mathrm{i}\partial\bar{\omega}$ *for* $\omega \in A^{1,0}(X)$; $\qquad d(*\omega) = -\mathrm{i}\bar\partial\omega$ *for* $\omega \in A^{0,1}(X)$.

f) $*\partial f = \mathrm{i}\bar\partial \bar{f}$; $\quad *\bar\partial f = -\mathrm{i}\partial \bar{f}$.

g) $d(*df) = 2\mathrm{i}\partial\bar\partial \bar{f}$.

The proof can be given by direct calculations with coordinates. □

The announced new characterization of harmonic differentials uses this star operator:

1.6 Proposition. *A differential ω on a Riemann surface is harmonic if and only if* Pdhs

$$d\omega = d(*\omega) = 0.$$

Proof. Let ω be harmonic. We show the two differential equations. Because of 1.3 we can assume that ω is holomorphic or antiholomorphic. But then $*\omega$ is antiholomorphic or holomorphic and the statement is clear.

Now we assume conversely the two differential equations. From $d\omega$ follows that ω can be locally written as df with some function f. A calculation in coordinates shows that $d*\omega = d(*df) = 0$ means that f is harmonic (use 1.5, g)). □

We denote by $\mathrm{Harm}^1(X)$ the space of all harmonic differentials on X. We reformulate 1.3 in the form

$$\mathrm{Harm}^1(X) = \Omega(X) \oplus \bar{\Omega}(X).$$

Hence for compact X we get

$$\dim_{\mathbb{C}} \mathrm{Harm}^1(X) = 2g.$$

2. Hodge theorie of compact Riemann surfaces

For two differentials α, β on a Riemann surface we define

$$[\alpha, \beta] = \alpha \wedge *\beta.$$

This is $\mathbb{C}$-linear in the first variable and satisfies

$$[\beta, \alpha] = \overline{[\alpha, \beta]}.$$

We express $[\alpha, \alpha]$ in local coordinates and write α for this purpose in the form

$$\alpha = f dz + g d\bar{z}.$$

Then

$$*\alpha = \mathrm{i}(\bar{f} d\bar{z} - \bar{g} dz)$$

and hence

$$[\alpha, \alpha] = \mathrm{i}(|f|^2 + |g|^2) dz \wedge d\bar{z} = 2(|f|^2 + |g|^2) dx \wedge dy.$$

The essential point here is that $(|f|^2 + |g|^2)$ is non-negative and zero only if the (smooth) functions f, g are zero.

Now we will assume that the Riemann surface X is compact. Then we can define

$$\langle \alpha, \beta \rangle = \int_X [\alpha, \beta].$$

This is a hermitian form on $A^1(X)$ ($\mathbb{C}$-linear in the first variable and with the property $\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$). Morover this form is positive definite. Hence $A^1(X)$ has been equipped with a structure as unitary vector space.

2.1 Proposition. *On any compact Riemann surface one has an orthogonal (with respect to $\langle\cdot,\cdot\rangle$) decomposition* Pned

$$A^{0,1}(X) = \bar\partial C^\infty(X) \oplus \bar\Omega(X).$$

Corollary. *A form σ in $A^{0,1}(X)$ can be written in the form $\sigma = \bar\partial f$ if and only if*

$$\int_X \sigma \wedge \omega = 0$$

for all holomorphic differentials ω.

Proof. Firstly we prove that $\bar\partial C^\infty(X)$ and $\bar\Omega(X)$ are orthogonal: We have to consider $\omega \wedge *\bar\partial f$ for antiholomorphic ω. Using the rules 1.5 one shows

$$\omega \wedge *\bar\partial f = -\mathrm{i}d(\bar f\omega).$$

From Stokes formula follows $\langle \omega, df\rangle = 0$. From the orthogonality we obtain that the natural map

$$\bar\Omega(X) \to A^{0,1}(X)/\bar\partial C^\infty(X)$$

is injective. Hence for the proof of 2.1 it remains to show that both sides have the same dimension. The dimension of $\bar\Omega$ is g (by duality). By Dolbeault's theorem V.6.3 the right hand side is isomorphic to $H^1(X,\mathcal{O}_X)$, which by definition has also the dimension g. This completes the proof of 2.1. □

We mention that by trivial reason the spaces $A^{1,0}(X)$ and $A^{0,1}(X)$ are orthogonal (since $dz \wedge dz = 0$). Hence not only $\bar\partial C^\infty(X)$ but also $\partial C^\infty(X)$ is orthogonal to $\bar\Omega(X)$. Hence also $dC^\infty(X)$ is orthogonal to $\bar\Omega/X)$. Since $dC^\infty(X)$ is invariant under complex conjugation it is also orthogonal to $\Omega(X)$. It follows that $dC^\infty(X)$ and $\mathrm{Harm}^1(X)$ are orthogonal. Let now u be a harmonic function on X. Then du is a harmonic differential which is orthogonal to $\mathrm{Harm}^1(X)$ hence to itself. This means $du = 0$ and we obtain:

2.2 Proposition. *Every harmonic function on a compact Riemann surface is constant.* Ehic

We give another application of 2.1. complex conjugation gives

$$A^{1,9}(X) = \partial C^\infty(X) \oplus \Omega(X)$$

or

$$A^1(X) = \partial C^\infty(X) \oplus \bar\partial C^\infty(X) \oplus \mathrm{Harm}^1(X).$$

Now we use the rule 1.5, f). It shows that

$$d\mathcal{C}^\infty(X) + *d\mathcal{C}^\infty(X) \subset \partial\mathcal{C}^\infty(X) + \bar\partial\mathcal{C}^\infty(X).$$

From the same rule follows also $df + *d(\mathrm{i}\bar f) = \partial f$ and hence the converse inclusion. The spaces $d\mathcal{C}^\infty(X)$ and $*d\mathcal{C}^\infty(X)$ are also orthogonal. This follows from $df \wedge **dg = -df \wedge dg = -d(fdg)$. This gives us

2.3 Proposition. *On a compact Riemann surface one has the orthogonal decomposition* OAec

$$A^1(X) = dC^\infty(X) \oplus *dC^\infty(X) \oplus \mathrm{Harm}^1(X).$$

We want to determine the position of $K := \mathrm{Kernel}(A^1(X) \xrightarrow{d} A^2(X))$ with respect to this position. We claim that it is orthogonal to $*dC^\infty(X)$. For $d\omega = 0$ one gets

$$\omega \wedge *(*df) = -\omega \wedge df = d(f\omega).$$

This shows $\langle \omega, *df\rangle = 0$. On the other side $dC^\infty(X)$ and $\oplus\mathrm{Harm}^1(X)$ are contained in K. Hence K equals their sum:

2.4 Proposition. *On a compact Riemann surface one has the orthogonal decomposition* OAed

$$\mathrm{Kernel}(A^1(X) \xrightarrow{d} A^2(X)) = dC^\infty(X) \oplus \mathrm{Harm}^1(X).$$

This proposition induces an isomorphism

$$\mathrm{Harm}^1(X) \cong \frac{\mathrm{Kernel}(A^1(X) \xrightarrow{d} A^2(X))}{dC^\infty(X)}.$$

By the theorem of de Rham the right hand side is isomorphic to $H^1(X,\mathbb{C})$. This gives us

2.5 Theorem of de Rham-Hodge. *For a compact Riemann surface one has* TRH

$$H^1(X,\mathbb{C}) \cong \mathrm{Harm}^1(X).$$

Hence the dimension of $H^1(X,\mathbb{C})$ is $2g$.

Notice that $H^1(X,\mathbb{C})$ depends only on the toplological space X and not on the complex structure. Hence we see that the genus g for homeomorphic Riemann surfaces is the same. But homeomorphic Riemann surfaces need not to be biholomorphic equivalent as already the example of tori shows.

3. Periods

Let ω be a (smooth) closed differential on a Riemann surface. A complex number C is called a *period* of ω, if there exists a closed curve α with the property

$$C = \int_\alpha \omega.$$

To explain the notion "period" we consider the case of a torus $X = \mathbb{C}/L$ and "$\omega = dz$". Every closed curve α in X lifts to a curve $\beta : [0,1] \to \mathbb{C}$. Then

$$\int_\alpha dz = \beta(1) - \beta(0).$$

Since α is closed we have $\beta(0) \equiv \beta(1) \bmod L$. Hence we have that the periods are precisely the elements of L.

It is sometimes necessary to choose a base point $a \in X$ and to consider only curves which start from a. If α is an arbitrary closed curve, one can combine a with $\alpha(0)$, then run through α and then go back the same way as one started. This shows that each period can be obtained by a curve starting and ending in a. Since the integral of closed differential is homotopy invariant we get a map

$$\pi_1(X, a) \longrightarrow \mathbb{C}, \quad \text{“}\alpha \longmapsto \int_\alpha \omega\text{''},$$

which is obviously a homomorphism.

The structure of the fundamental group of a compact surface can be determined by topological methods. A result, which can be obtained without any further theory is

3.1 Lemma. *Let $S \subset X$ be a finite subset of a compact Riemann surface, $a \in X - S$. The fundamental group $\pi_1(X, a)$ is countable.* Fgic

Proof. If $X = \bar{\mathbb{C}}$ this can be seen as follows. Take the base point a to be rational. This means that real and imaginary part are rational numbers. It is easy to see that each closed curve with origin a is homotopic to piecewise linear curve with rational vertices. This is a countable set of curves.

The general case can be settled similarly choosing a non-constant meromorphic function $f : X \to \bar{\mathbb{C}}$. Then we consider curves in X, which map under f to piece wise linear curves with rational vertices as considered in the first case. This is a countable set of curves and the same argument as in the first case works. □

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The importance of the periods shows the following

3.2 Remark. *A closed differential ($d\omega = 0$) is total, i.e. of the form $\omega = df$ if and only if all its periods vanish.* AcdT

Proof. The main theorem of calculus says

$$\int_\alpha df = f(\alpha(1)) - f(\alpha(0)).$$

This shows that the periods of df vanish. To prove the converse we *define*

$$f(x) := \int_a^x \omega.$$

Here a is a fixed chosen base point. The integral is understood as a curve integral along a curve which connects a with x. Since the periods vanish this integral is independent of the choice of this curve. A local computation shows that $df = \omega$. □

Harmonic differentials are closely tied to their periods:

3.3 Proposition. *A harmonic differential on a compact Riemann surface vanishes if all its periods are zero.* AhdV

Proof. Assume that ω is a harmonic differential whose periods vanish. Then $\omega = df$ by 3.2. Now 2.3 shows that ω is orthogonal to $\mathrm{Harm}^1(X)$ hence to itself.This shows $\omega = 0$. □

A variant of 3.3 states:

3.4 Proposition. *A holomorphic differential on a compact Riemann surface vanishes if the real parts of all its periods are zero.* AhdVz

Proof. From 3.3 follows that $\mathrm{Re}\,\omega = (\omega+\bar\omega)/2$ is zero. Locally ω can be written as df with a holomorphic function. It follows that f has constant real part. But then f is constant and $\omega = 0$. □

We now want to consider the periods of *all* holomorphic differentials together. This leads to the following

3.5 Definition. *A $\mathbb{C}$-linear form $l : \Omega(X) \to \mathbb{C}$ is called a* ***period*** *of the compact Riemann surface X, if the exists a closed curve α with the property* DaP

$$l(\omega) = \int_\alpha \omega.$$

The set L of all periods is a subset of the dual space $\Omega(X)^*$. Actually it is an additive subgroup $L \subset \Omega(X)^*$. Hence we can consider the factor group

$$\mathrm{Jac}(X) := \Omega(X)/L.$$

We will show later that L is a lattice and therefore $\mathrm{Jac}(X)$ a torus of real dimension $2g$. This will be explained in detail. Here we take its just as justification to call $\mathrm{Jac}(X)$ the *Jacobian variety.* We only mention that from 3.1 follows that L is a countable set.

3.6 Definition of the Abel-Jacobi map. *Let X be a Riemann surface with a base point a. Then the Abel-Jacobi map* DAJ

$$A : X \longrightarrow \operatorname{Jac}(X)$$

is defined as follows. For a point $x \in X$ one chooses a curve α which combines a with x. Then one considers the linear form

$$\omega \longmapsto \int_a^x \omega := \int_\alpha \omega.$$

The image of this linear form in $\operatorname{Jac}(X)$ is independent of the choice of α and is defined to be $A(x)$.

The Abel-Jacobi map admits certain important variants, which we will all denote by the same letter A: For example one can consider for any natural number d the d-fold cartesian product $X^d := X \times \ldots \times X$ and define

$$A : X^d \longrightarrow \operatorname{Jac}(X), \qquad A(x_1, \ldots, x_d) = A(x_1) + \cdots + A(x_d).$$

Here we used that $\operatorname{Jac}(X)$ carries a natural structure as abelian group.

A closely related extension is obtained as follows. Consider a divisor D on X. The we define

$$A(D) := \sum_{a \in X} D(x) A(x).$$

This is obviously a homomorphism

$$A : \operatorname{Div}(X) \longrightarrow \operatorname{Jac}(X).$$

Of course this homomorphism depends on the choice of the base point a. But now we restrict A to the subgroup $\operatorname{Div}^0(X)$ of divisors if degree 0. Then we get:

3.7 Remark. *The Abel-Jacobi map $A : \operatorname{Div}^0(X) \longrightarrow \operatorname{Jac}(X)$ restricted to the divisors of degree zero is independent of the choice of the base point.* AJib

Proof. We take another base point and choose a fixed curve β which combines a and b. We use this curve to transform closed curves with origin a to curves with origin b. Then the two Abel-Jacobi map differ by

$$\sum_{x \in X} D(x) \int_\beta \omega.$$

This is zero since D has degree zero. □

The Abel Jacobi map on $\operatorname{Div}^0(X)$ can be defined in a slightly different way:

3.8 Remark. *Let $D = (a_1) + \cdots + (a_n) - (b_1) - \cdots - (b_n)$ be a divisor (of degree zero) on the compact Riemann surface X. Let γ_i be curves which join a_i with b_i. Then $A(D)$ is represented by the linear form* DrbLf

$$\omega \longmapsto \sum_{i=1}^{g} \int_{\gamma_i} \omega.$$

Proof. One chooses a base point and joins a and a_i by curves α_i. Then one defines curves from a to b_i by joining α_i and γ_i. □

Special divisors of degree 0 are the principal divisors (f). Recall that the define a subgroup $\mathcal{H}(X) \subset \mathrm{Div}^0(X)$. The factor group

$$\mathcal{D}^0(X)/\mathcal{H}(X)$$

can be identified with a subgroup $\mathrm{Pic}^0(X)$ of the Picard group (VI.3.1). It can be considered as the group of all isomorphy classes of line bundles of degree 0.

3.9 Proposition. *The image of a prinicipal divisor in* $\mathrm{Jac}(X)$ *under the Abel-Jacobi map is zero. Hence it induces a homomorphism* IopDz

$$A : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Jac}(X).$$

Proof. Let f be a non-constant meromorphic function on the compact Riemann surface X. We consider it as a map $f : X \to \bar{\mathbb{C}}$. For each $z \in \bar{\mathbb{C}}$ we consider the fibre $D_z := f^{-1}(z)$. Recall that we can talk about the multiplicity of a point with value a. Hence D_z can be considered naturally as a divisor. We will show that all D_z have the same image in $\mathrm{Jac}(X)$. (This proves 3.9, since $(f) = D_0 - D_\infty$.) We have to show that the map

$$\bar{\mathbb{C}} \longrightarrow \mathrm{Jac}(X), \quad z \longmapsto A(D_z),$$

is constant. What we will actually prove, that this map is *locally liftable.*

A map $h : \bar{\mathbb{C}} \longrightarrow$ $\mathrm{Jac}(X)$ *is called locally liftable, if every point of $\bar{\mathbb{C}}$ admits an open neighborhood U, such that $h|U$ lifts to a holomorphic map $H : U \to \Omega(X)^*$.*

It should be clear what "holomorphic map" here means. For example one can say that $H(z)(\omega)$ is holomorphic in the usual sense for every $\omega \in \Omega(X)$. Before we proof the lifting property we show that it will solve our problem:

3.10 Lemma. *Every locally liftable map* $h : \bar{\mathbb{C}} \to \mathrm{Jac}(X)$ *is constant.* Ellc

Proof of the Lemma. We fix a form $\omega \in \Omega(X)$ For a local lift H defined on some U we consider the holomorphic differential $\omega_H = d(H(z)(\omega))$ on U. Let G be a local lift on some other V. Then we have on the intersection

$$H(z) = G(z) + \lambda(z), \quad \lambda(z) \in L.$$

The function $\lambda(z)$ is a holomorphic function whose values lie in the countable set L (use 3.1). This means that λ is locally constant. So ω_H and ω_G coincide on the intersection. Hence they glue to a holomorphic differential on $\bar{\mathbb{C}}$. Since every holomorphic differential on the sphere is zero, we get that h is locally constant. Since $\bar{\mathbb{C}}$ is connected we get that h is constant. □

Proof of 3.9 continued. It remains to show that the map $h : \bar{\mathbb{C}} \to \mathrm{Jac}\, X$, $h(z) = D_z$, is locally liftable. We fix a point $b \in \bar{\mathbb{C}}$ and investigate the map h close to b. Let $b_1, \dots, b_m$ be the points in X over a with multiplicities $k_1, \dots, k_m$. Then $n = k_1 + \cdots + k_m$ is the covering degree of f. For the Abel-Jacobi map we need a base point $a \in X$ and curves $\alpha_1, \dots, \alpha_n$ which combine a with the b_i. This is understood as follows. The first m_1 curves go from a to b_1, the next m_2 from a to b_2 and so on. We consider small disc U around b. We know that then $f^{-1}(z)$ is the union of disjoint discs $U_1, \dots, U_m$ discs such that $b_i \in U_i$ for $1 \le i \le m$. Now take a point $z \in U$ which is different from a. Since U can been taken small enough the point z will have n inverse images, where n is the covering degree of f. The n points are distributed over the m discs $U_1, \dots, U_m$. We can assume that the first m_1 are in U_1, the next m_2 in U_2 and so on. Now consider curves β_i, $1 \le i \le n$ as follows. The first m_1 ones lie in U_1 and combine b_1 with the z-s which lie in U_1. The next m_2 ones lie in U_2 ans combine b_2 with the z-s which lie in U_2 and so on. Now we consider the sum of the integrals

$$\int_{\beta_i} \omega \qquad (\omega \in \Omega(X)).$$

It is independent of the choice of the β_i and depends holomorphically on z. This sum represents the difference of $A(D_b) - A(D_z)$. We can take this sum to get a local lifting of h. □

4. Abel's theorem

Abels's theorem states that the map $A : \mathrm{Pic}^0(X) \to \mathrm{Jac}(X)$ is injective. To prove this, we need some local preperation. We consider the unit disc E.

4.1 Definition. *A* $\mathcal{C}^\infty$*-function* f *on* $E - \{0\}$ *has order* k *in* 0*, if* $g(z) = f(z) \cdot z^{-k}$ *extends to a* $\mathcal{C}^\infty$*-function on* E*, which has no zero in* 0. DslE

We are interested in the logarithmic derivative

$$\frac{df}{f} = k\frac{dz}{z} + \frac{dg}{g}.$$

Notice that dg/g is smooth on the whole E. Let ω be a differential on E with compact support. We want to consider the integral

$$\int_E \frac{df}{f} \wedge \omega.$$

Since df/f is singular at 0, we have to check the existence of this integral. We write $\omega = h_1 dz + h_2 d\bar{z}$. Then

$$\frac{dz}{z} \wedge \omega = \frac{h_1}{z} dz \wedge d\bar{z} = 2\mathrm{i}\frac{h_1}{z} dx \wedge dy.$$

To compute the integral we use polar coordinates $z = re^{\mathrm{i}\varphi}$. Then

$$\int_E \frac{dz}{z} \wedge \omega = 2\mathrm{i} \int_0^1 \int_0^{2\pi} \frac{h_1(re^{\mathrm{i}\varphi})}{re^{\mathrm{i}\varphi}} r dr d\varphi.$$

Since the factor r cancels, there is no problem with the existence of the integral.

Let g be a $\mathcal{C}^\infty$-function with compact support. Then

$$\frac{df}{f} \wedge dg = d\Big(g \cdot \frac{df}{f}\Big).$$

We want to integrate over a circle $|z| = r$, where $r < 1$ has been chosen so close to 1 such that g vanishes for $|z| \geq r$. Since gdf/f has a singularity in the origin, we have to be careful with the application of the Stokes formula. We have to choose a small $\varepsilon > 0$ and then we can say

$$\int_E \frac{df}{f} \wedge dg = -\oint_{|z|=\varepsilon} g \cdot \frac{df}{f}.$$

(We take the integral in the mathematical positive sense, hence we need a minus sign.) We will take the limit $\varepsilon \to 0$. Then we can replace g by the constant $g(0)$ and f by z^k. This gives:

4.2 Lemma. *Let f be a $\mathcal{C}^\infty$-function on the punctured disk $E - \{0\}$, which is of order k at the origin. Let g be a $\mathcal{C}^\infty$-function on E with compact support. Then* LfoD

$$\int_E \frac{df}{f} \wedge dg = kg(0).$$

The definition of order 4.1 can be generalized to Riemann surfaces in an obvious way: If f is a differentiable function in a punctured neighborhood of a point a of a Riemann surface X, one chooses a disc $\varphi : U \to E$, $\varphi(a)0$ around this point inside this neighborhood. Then f has order k in a if f_φ has order k in 0. It is clear that this definition doesn't depend on the choice of the disc.

4.3 Definition. *Let D be a divisor on a Riemann surface X. By a weak solution of D one understands a differentiable function in the complement of the support of D, such that f has order $D(a)$ for each a of the support.* DwS

Lemma 4.2 has then the obvious generalization:

4.4 Lemma. *Let f be a weak solution of a divisor D on a Riemann surface X and let g be a differentiable function on X with compact support. Then* LfoDz

$$\frac{1}{2\pi \mathrm{i}} \int_X \frac{df}{f} \wedge dg = \sum_{a \in X} D(a) g(a).$$

(The sum is finite.)

Next we have to construct weak solution. The essential part is a local construction:

4.5 Lemma. *Let a, b be points of the unit disc E. There is a weak solution f of the divisor $(b) - (a)$ with the following additional property. There exists $0 < r < 1$ such that $f(z)$ is identical one for $r \leq |z| < 1$.* LlesL

Proof. In the case $a = b$ we can take $f \equiv 1$, hence we can assume $a \neq b$. We need a simple but basic observation from complex calculus. The function $(z-1)/(z+1)$ takes values on the negative real axis of and only if z is contained in the interval $[-1, 1]$. Hence the principal value of the logarithm defines a *holomorphic* function

$$\mathbb{C} - [-1, 1] \longrightarrow \mathbb{C}, \quad z \longmapsto \log \frac{z-1}{z+1}.$$

An easy consequence is:

Let a, b be two different points in the disc $|z| \leq r$. Then there exists in the complement $|z| > r$ a holomorphic branch of the logarithm $\log \frac{z-b}{z-a}$.

In our situation we can take $r < 1$. Now we consider a differentiable function ψ on [r,1] which is 1 close to r and 0 close to 1. The function

$$\exp\Bigl(\psi \log \frac{z-b}{z-a}\Bigr) \qquad r \leq |z| < 1$$

is 1 if $|z|$ is close to 1 and $\frac{z-a}{z-b}$ if $|z|$ is close to r. Hence we can glue it with the function

$$\frac{z-b}{z-a} \qquad |z| \leq r \ (z \neq a, b).$$

This gives the proof of 4.5. □

Again there is a generalization to Riemann surfaces:

4.6 Lemma. *Let $\alpha : [0,1] \to X$ be a curve on a Riemann surface, $a = \alpha(0)$, $b = \alpha(1)$. There exists a weak solution of the divisor $(b)-(a)$ with the following properties:* LcRsw
1. *The function f is constant one outside some compact set.*
2. *For every closed differential ω on X one has*

$$\int_\alpha \omega = \frac{1}{2\pi \mathrm{i}} \int_X \frac{df}{f} \wedge \omega.$$

Proof. First we mention that df vanishes outside a compact set. Hence the existence of the integral on the right hand side is only a local problem around the singularities a and b, which we settled above.

Let $c = \alpha(t_0)$, $0 \le t_0 < 1$ be another point on the curve. Assume that the problem has been solved for the curve $\alpha|[0,t_0]$. Denote the solution by f_1. Let similarly f_2 be a solution for the curve $\alpha|[t_0,1]$. Then $f_1 f_2$ gives a solution for the total curve α. This observation allows us to restrict to the case, where the curve is contained in a disc U. In this disk we get a weak solution from 4.5. This solution can be extended by one to a solution on X. Now we have to prove the integral formula. In the disc U the differential ω is total, $\omega = dg$. Now we take a smaller disc $V \subset\subset U$ such that f is one outside V. We can modify g to get a differentiable function with compact support such that the equation $\omega = dg$ still holds on V. Then

$$\int_\alpha \omega = g(b) - g(a)$$

and by 4.4 we also have

$$\frac{1}{2\pi \mathrm{i}} \int_X \frac{df}{f} \wedge \omega = g(b) - g(a). \qquad \square$$

Now we can see that every divisor of degree zero on a compact Riemann surface admits a weak solution. We write the divisor in the form

$$D = (a_1) + \cdots + (a_n) - (b_1) - \cdots - (b_n).$$

We join a_i and b_i by a curve γ_i. Using 4.6 we find a weak solution f of of the divisor D which has the additional property

$$\sum_{i=1}^n \int_{\alpha_i} \omega - \sum_{i=1}^n \int_{\beta_i} \omega = \frac{1}{2\pi \mathrm{i}} \int_X \frac{df}{f} \wedge \omega$$

for all closed differentials ω. We will use this for *holomorphic* ω. Then

$$\frac{df}{f} \wedge \omega = \frac{\bar\partial f}{f} \wedge \omega.$$

Recall that f locally is of the form $z^k g(z)$ with a differentiable function g without zero. This implies (locally)

$$\frac{\bar\partial f}{f} = \frac{\bar\partial g}{g}.$$

Hence

$$\sigma := \frac{\bar\partial f}{f} \in A^{0,1}(X)$$

is differentiable everywhere.

Let's assume now that the divisor D is in the kernel of the Abel-Jacobi map. Then we get

$$\frac{1}{2\pi\mathrm{i}}\int_X \sigma\wedge\omega = \sum_{i=1}^n \int_{\gamma_i}\omega = 0$$

for all holomorphic ω. But this implies $\sigma = \bar\partial h$ (corollary of 2.1). We use h to modify the weak solution f:

$$F = e^{-h} f.$$

We obtain

$$\bar\partial F = -e^{-g} f\bar\partial h + e^{-h}\bar\partial f = 0.$$

Hence F is a meromorphic solution. This gives

4.7 Abel's theorem. *The homomorphism* AT

$$A : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Jac}(X)$$

is injective.

In other words: A divisor of degree zero is principal if and only if its image in $\mathrm{Jac}(X)$ *is zero.*

As an example we treat a torus $X = \mathbb{C}/L$. Recall that $\Omega(X)$ is generated by "$\omega = dz$". We represent the divisor D of degree zero by a divisor $(a_1)+\cdots+(a_n)-(b_1)-\cdots-(b_n)$ in the interior of some fundamental parallelogram. We join a_i and b_i by curves γ_i inside this interior Since

$$\sum_{i=1}^n \int_{\gamma_i} dz = \sum_{i=1}^n b_i - \sum_{i=1}^n a_i.$$

This gives the well-known Abel theorem for elliptic functions:

A divisor of degree zero $(a_1)+\cdots+(a_n)-(b_1)-\cdots-(b_n)$ *on a torus is principal if and only if* $a_1+\cdots+a_n-b_1-\cdots-b_n = 0$.

5. The Jacobian inversion problem

All what we have proved about the group $L \subset \Omega(X)^*$ is that it is countable group. We want to prove more, namely that it is a lattice. By a lattice of a finite dimensional real vector space V we understand a subgroup L of the form

$$L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$$

where $e_1, \dots, e_n$ is a basis of V. A lattice is obviously a discrete subgroup which generates V as vector space. The converse is also true. We will use this without proof.

By a lattice of a finite dimensional complex vector space we understand a lattice of the underlying real vector space.

5.1 Proposition. *The set of periods $L \subset \Omega(X)^*$ of a compact Riemann surfaces is a lattice. Hence* $\mathrm{Jac}(X)$ *is a torus of real dimension* $2g$. SpiL

First part of the proof: L is discrete.

Let V be a complex vector space of dimension g. If $\mathfrak{M}$ is a set of sub vector spaces whose intersection is 0, then there exist g vector spaces in $\mathfrak{M}$ whose intersection is zero. For a point $a \in X$ we can consider the subspace of all $\omega \in \Omega(X)$, which vanish in a. The above remark shows, that there exist g points $a_1, \dots, a_g$ such that a form $\omega \in \Omega(X)$ vanishes if it vanishes in these points. Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. We choose discs $z_i : U_i \to E$ around the points a_i. In these charts $\omega_i = f_{ij} dz_j$. Here we write f_{ij} as a function on U_i and not on E. Then

$$A = \big(f_{ij}(a_j)\big)_{1 \le i,j \le g}$$

is an invertible matrix. In the disc U_i and we define the holomorphic function

$$F_i(x) = \int_{a_i}^{x} \omega_i,$$

where the integral is taken along a path from a_j to x inside U_i. then we consider the map

$$F : U_1 \times \cdot \times U_g \longrightarrow \mathbb{C}^g, \quad F(x_1, \dots, x_g) = F_1(x_1) + \cdots + F_g(x_g).$$

The (complex) Jacobian of F with respect to the charts is (f_{ij}). From the theorem of invertible functions follows that

$$W := F(U_1 \times \cdots \times U_g) \subset \mathbb{C}^g$$

is a neighborhood of $F(a_1, \dots, a_g)$.

Here we use a complex version of the theorem of invertible functions of several variables. But this is a consequence of the real version. The point is that F is differentiable in the real sense and the real functional determinant is the square of the absolute value of the complex functional determinant. To prove this one has to express the real derivatives by the complex ones and then use the Cauchy Riemann equations.

The basis ω_1, ω_g induces a dual basis $\omega_1^*, \dots, \omega_g^*$ of $\Omega(X)^*$, namely

$$\omega_i^*(\omega_j) = \delta_{ij}.$$

We use this basis to identify the space $\Omega(X)^*$ with $\mathbb{C}^g$. Then the map F can be read as a map

$$F : U_1 \times \cdots \times U_g \longrightarrow \Omega(X)^*.$$

The advantage is that it doesn't depend on the choice of the basis, since

$$F(x_1, \dots, x_g)(\omega) = \sum_{i=1}^{g} \int_{a_i}^{x_i} \omega.$$

(We used the coordinates just since the theorem of invertible functions often is not formulated in a coordinate invariant manner.)

We claim the the neighborhood $0 \in W$ has empty intersection with L. (Since L is a group this implies the discreteness of L.) We argue by contradiction and assume $F(x_1, \dots, x_g) = 0$ Since $F(x_1, \dots, x_g)$ represents the image of the divisor $D = (x_1) + \cdots + (x_g) - (a_1) - \cdots - (a_g)$ under the Abel-Jacobi map, we can apply Abels's theorem 4.7. There exists a meromorphic function f with $(f) = D$. The function f has poles of order one in the a_i. We denote by $C_i \neq 0$ the residue of f with respect to the chart z_i. Then

$$\mathrm{Res}_{a_i}(f\omega_i) = C_i f_{ij}(a_j).$$

From the residue theorem applied to the meromorphic differentials $f\omega_i$ follows

$$\sum_{j=1}^{g} C_j \varphi_{ij}(a_j) = 0.$$

But then the matrix A cannot be invertible, which gives a contradiction.

Second part of the proof: L generates $\Omega(X)^*$ as real vector space.

We have to show that a real linear form L on $\Omega(X)^*$, which vanishes on L is zero. Every real linear form is the real part of a $\mathbb{C}$-linear complex linear form. Every complex linear form on $\Omega(X)^*$ is of the form

$$l \longmapsto l(\omega)$$

for some $\omega \in \Omega(X)$. The real part of this linear form vanishes on L if and only if the real parts of the periods of ω are zero. But then ω is zero (3.4). □

The first part of the proof of 5.1 shows that a full neighbourhood of $0 \in \mathrm{Jac}(X)$ is contained in the image of the Abel-Jacobi map. The image is also a group. Obviously a torus is generated as group by any neighborhood of the origin. Hence we obtain that the Abel-Jacbobi map is not only injective but also surjective.

5.2 Theorem. *The Abel-Jacobi map* AJii

$$A : \mathrm{Pic}^0(X) \stackrel{\sim}{\longrightarrow} \mathrm{Jac}(X)$$

is an isomorphism.

We com back to the Abel-Jacobi map in the form

$$A : X^d \longrightarrow \mathrm{Jac}(X).$$

It depends on the choice of a base point a.

5.3 Remark. *Let X be compact Riemann surface of genus $g > 1$. Then* cRiJ

$$A : X \longrightarrow \mathrm{Jac}(X)$$

is injective.

Proof. We mention that a bijective holomorphic map $f : X \to Y$ between Riemann surfaces is biholomorphic. This is known from complex calculus in the case of open domains in $\mathbb{C}$. The general case works in the same way. Let f be a meromorphic function on a compact Riemann surface with divisor $(b)-(a)$. The it is of order one and hence defines a bijective map $X \to \bar{\mathbb{C}}$. Hence X is biholomorphic equivalent to $\bar{\mathbb{C}}$. This implies 5.3. □

We call a point $a \in X^g$ *generic*, if every holomorphic differential ω, which vanishes at all points occurring in a, vanishes identically. In the fist part of the proof of 5.1 we have shown that there are generic points. The argument shows a little more, namely:

5.4 Remark. *The set of generic points is open and dense in X^g. The map $A : X^g \longrightarrow \mathrm{Jac}(X)$ is locally topological on the set of generic points.* SgpT

Let now X be a compact Riemann surface of genus one. Then $\mathrm{Jac}(X) = \mathbb{C}/L$ is also a Riemann surface. The map $A : X \to \mathrm{Jac}(X)$ is injective. Since it is also injective we obtain:

5.5 Theorem. *Every compact Riemann surface of genus one is biholomorphic equivalent to a torus $\mathbb{C}/L$.* cRsT

Now we switch to $A : X^d \to \mathrm{Jac}(X)$. One may ask, whether this map is surjective for suitable d. Since the real dimension of X^d is $2d$ and of $\mathrm{Jac}(X)$ is $2g$, one can hope, that this is true for $d = g$.

5.6 Theorem. *The Abel-Jacobi map* AJS

$$A : X^g \longrightarrow \mathrm{Jac}(X)$$

is surjective.

Proof. let $a \in X$ be the base point. Since $\mathrm{Pic}^0(X) \to \mathrm{Jac}(X)$ is surjective, we only must show the following:

Every divisor D of degree 0 is quivalent to a divisor $(b_1) + \cdots + (b_g) - g \cdot (a)$.

For the proof we consider the divisor $D' = D + g(a)$. It has degree g. Riemann-Roch implies that $\dim \mathcal{O}_D(X) \geq 1$. Hence there exists a non-zero meromorphic function f such that $(f) + D' \geq 0$. Since the degree of this divisor is g we get $(f) + D' = (b_1) + \cdots + (b_g)$. This gives $(f) + D = (b_1) + \cdots + (b_g) - g(a)$. □

The symmetric group S_g acts on X^g by permutation of the components. The quotient

$$X^{(g)} = X^g / S_g$$

can be considered as the the set of unordered n-tuples of X. Since the map $X^g \to \mathrm{Jac}(X)$ does not depend on the ordering of the points, it factors through the natural projection $X^g \to X^{(g)}$. We obtain the *Jacobi map*

$$J : X^{(g)} \longrightarrow \mathrm{Jac}(X).$$

The Jacobi inversion problem asks for the inversion of this map. Actually for $g > 1$ this map is not bijective but it is close to a bijective map. The correct statement is:

5.7 Jacobi inversion theorem, unprecise statement. *The map* JIT

$$J : X^{(g)} \longrightarrow \mathrm{Jac}(X).$$

is ***bimeromorphic.***

Up to no we didn't define what bimeromorphic means. This will be part of what follows.

6. The fibres of the Jacobi map

We can identify the elements of $X^{(g)}$ with divisors $D \geq 0$ of degree 0. We have to determine the fibre $J^{-1}(J(D))$. It consists of all divisors $D' \geq= 0$ of degree g which are equivalent to D. Then there exist a meromorphic function with $D' = D + (f)$. Because of $D' \geq 0$ we have $f \in \mathcal{O}_D(X)$. The function f is determined up to a constant factor. Hence it is better to identify two f if they are contained in the same one-dimensional complex sub-vector space. Recall that the set of all one-dimensional sub-vector spaces of a vector space V is called the projective space $P(V)$. There is a natural map $V - \{0\} \to P(V)$. We use this (for finite dimensional V) to define a topology on $P(V)$, the quotient topology. It is easy to show that this is a compact space.

6.1 Remark. *Let $D \in X^{(g)}$. There is a natural bijective map* mfiS

$$P(\mathcal{O}_D(X)) \xrightarrow{\sim} f^{-1}(f(D)), \quad \text{“}f \longmapsto D + (f)\text{''}.$$

The space X^g carries the product topology and $X^{(g)} = X^g/S_g$ the quotient topology. It is easy to see that this is a Hausdorff space.

6.2 Lemma. *The map* foJc

$$P(\mathcal{O}_D(X)) \xrightarrow{\sim} f^{-1}(f(D)), \quad f \longmapsto D + (f),$$

is topological.

Corollary. *The fibres of the Jacobi map are connected.*

From 5.4 we know that a generic point in X^g is isolated in the fibre of $X^g \to \mathrm{Jac}(X)$. As a consequence its image in $X^{(g)}$ is isolated in its fibre as well. From 6.2follows that is the only point in its fibre. Hence we obtain:

6.3 Theorem. *The Jacobi map $J : X^{(g)} \to \mathrm{Jac}(X)$ is surjective. There exists an open and dense subset of $X^{(g)}$, on which J is injective.* TJmi

This can be considered as a weak version of the Jacobi inversion theorem, which states that J is bimeromorphic.

7. The Jacobian inversion problems and abelian functions

We need some basic facts about holomorphic functions in several variables. A function $f : D \to \mathbb{C}^m$, where $D \subset \mathbb{C}^n$ is called complex differentiable in a point $a \in D$ if there exists a $\mathbb{C}$-linear map $J(f,a) : \mathbb{C}^n \to \mathbb{C}^m$, such that

$$f(z) - f(a) = J(f,a)(z-a) + r(z), \qquad \lim_{z \to a} \frac{r(z)}{z-a} = 0.$$

Hence complex differentiable implies real differentiable. The usual permanent properties including the theorem of invertible functions carry over to the complex case. The function f is called holomorphic if and only if it is complex differentiable in each point. This is the case if and only if every component of f is holomorphic. There is a basic

7.1 Lemma. *Let $f : D \to \mathbb{C}$ be a continuous function on an open domain $D \subset \mathbb{C}^n$ such that f is holomorphic in each variable separately. Then f is holomorphic. Moreover for every point $a \in D$ there exists a power series, which converges absolutely and locally uniform in some ball $U_r(a) \subset D$ and represents there f:* fcfP

$$f(z) = \sum_{\nu \in \mathbb{N}_0^n} (z-a)^\nu \qquad (a \in U_r(a)).$$

Here we use the usual writing with multiindices. The proof uses a simple generalization of the Cauchy integral formula. Applying the usual one in each variable one obtains:

$$f(z) = \frac{1}{(2\pi \mathrm{i})^n} \oint_{|\zeta_1 - a_1| = r} \cdots \oint_{|\zeta_n - a_n| = r} \frac{f(\zeta) d\zeta_1 \ldots d\zeta_n}{(z_1 - \zeta_1) \cdots (z_n - \zeta_n)}$$

Now the same argument as in the case $n = 1$ works: One expands the integrand in to a (multivariable) geometric series and interchanges summation and integration.

The power series expansion shows a weak form of the principle of analytic continuation: If two analytic functions $f, g : D \to \mathbb{C}^m$ on a connected open subset $D \subset \mathbb{C}^n$ agree on some non empty open subset then the agree everywhere.

Analytic manifolds

Analytic manifolds of complex dimension $n \geq 1$ are the straight forward generalization of Riemann surfaces ($n = 1$). Hence we can keep short:

An analytic manifold is a geometric space, which is locally isomorphic to a space $(U, \mathcal{O}_U)$, where $U \subset \mathbb{C}^n$ is an open subset and $\mathcal{O}_U$ is the sheaf of holomorphic functions. An analytic map between analytic manifolds is just a morphism of geometric spaces.

Basis facts of Riemann surfaces carry over to analytic manifolds. We mention just the following weak form of the principle of analytic continuation which easily follows by means of power series:

Let $f, g : X \to X$ be two analytic maps between to analytic manifolds. Assume that X is connected and that f and g agree on some non-empty open subset. Then f and g agree on the whole X.

Meromorphic functions

Zero sets of analytic functions are not discrete in general as for example the function $z_1 \cdot z_2$ on $\mathbb{C}^2$ shows. This makes the notion of a meromorphic function more delicated in the higher dimensional case. An example of a meromorphic function on $\mathbb{C}^2$ should be z_1/z_2. There is now clean way to define a value for this function at the origin 0. Hence it would be false to define meromorphic functions simply as holomorphic maps into $\bar{\mathbb{C}}$. This works only in the case $n = 1$. We proceed as follows:

Let X be an analytic manifold. We consider pairs (U, f), where $U \subset X$ is an open an dense subset of X and $f : U \to \mathbb{C}$ is an analytic function. We call this pair meromorphic on X, if for every point $a \in X$ (only $a \notin U$ is of interest), there exists a small open connected neigbourhood U and two analytic functions $g, h : U \to \mathbb{C}$, such that h is not identically zero and such that

$$f(x) = \frac{g(x)}{h(x)} \quad \text{for all} \quad x \in U,\ h(x) \neq 0.$$

two meromorphic pairs (U, f) and (V, g) are called equivalent, if f and g agree on $U \cap V$. A *meromorphic function* on X is a full equivalence class of such pairs. It is easy to define the sum and product of two meromorphic functions. It is also not difficult to prove that for connected X the set of all meromorphic functions is a field. (We don't know that U is connected. But this deosn't matter, since the condition of meromorphicity concerns the whole domain D.) We denote this field by $K(X)$.

Examples of analytic manifolds

The first example is a torus $X_L := \mathbb{C}^n/L$, where L is a lattice. As in the case $n = 1$ the meromorphic functions on X_L correspond uniquely to the L-periodic meromorphic functions on $\mathbb{C}^n$. Sich functions are called "*abelian functions.* They generalize the elliptic functions.

The direct product $X \times Y$ of two analytic manifolds carries also a structure as analytic manifold. As a consequence the power $X^n = X \times \ldots \times X$ of a Riemann surface is an analytic manifold.

We go back to the Abel Jacobi map

$$A : X^g \longrightarrow \operatorname{Jac}(X)$$

. Pulling back a function we get an imbedding of fields

$$K(\operatorname{Jac}(X)) \longrightarrow K(X^g).$$

The image is contained in the subfield $K(S^g)^{S_g}$ of all function which are invariant under arbitrary permutations.

It can be shown that The symmetric power $X^{(g)}$ is an analytic manifold as well and that $K(X^{(g)})$ can be naturally identified with $K(X^g)^{S_g}$. We will not prove this and take $K(S^g)^{S_g}$ just as substitute for the correct field $K(X^{(g)})$.

7.2 Jacobi inversion theorem, precise statement. *The natural map* JITz

$$K(\mathrm{Jac}(X)) \longrightarrow K(X^g)^{S_g}$$

is an isomorphism of fields.

We will not prove this here.

We work out a special case of 7.7.2, which will lead us to a formulation which is close to what hat Jacobi in mind. Let $f: X \to \bar{\mathbb{C}}$ be a non constant meromorphic function. It induces an analytic map

$$f^g : X^g \to \bar{\mathbb{C}}^g.$$

The projections give g analytic maps

$$f_\nu : X^g \longrightarrow \bar{\mathbb{C}} \qquad (1 \le \nu \le g).$$

They are just defined by

$$f_\nu(x_1, \dots, x_g) = f(x_\nu).$$

They can be considered as meromorphic functions on X^g. But they are not invariant under S_g. Hence we consider the elementary symmetric expressions

$$E_k = \sum_{1 \le \nu_1 \le \ldots \le \nu_k \le g} f(x_{\nu_1}) \ldots f(x_{\nu_k})$$

They are also meromorphic functions on X^g with the advantage to be symmetric. The Jacobi inversion theorem predicts the **existence of abelian functions** F_ν on X_L whose pull-back are the E_k.

Historical cases

We consider the Riemann surface X of the function $\sqrt{P(z}$ where P is a polynomial of degree 3 or 4 without multiple zero. Recall that the corresponding Riemann surface has genus one. Up to a finite number of points it is the curve $(z, w),\ w^2 = P(z)$. The natural projection $(z, w) \mapsto z$ is a meromorphic function on X, which we take for f. Recall that $dz/\sqrt{P(z)}$ generates $\Omega(X)$. The Abel-Jacobi map is

$$X \longrightarrow \mathbb{C}/L, \quad x \longmapsto \int_a^x \frac{dz}{\sqrt{P(z)}}.$$

The inversion theorem says in this case, that the function z is the pull-back of an elliptic function φ. Pull-map means that the composition of φ with the Abel-Jacobi map gives x, i.e.

$$x = \varphi\Big(\int_a^x \frac{dz}{\sqrt{P(z)}}\Big).$$

This reflects the classical observation of Abel that the inversion of the elliptic integral gives an elliptic function.

The big question was how to generalize this to the hyperelliptic case. Let's consider for the case $\sqrt{P(z)}$, where P now is of degree 5 ore 6 without multiple zero. Recall that a basis of $\Omega(X)$ in this case is given by

$$\frac{dz}{\sqrt{P(z)}}, \quad \frac{zdz}{\sqrt{P(z)}}.$$

The Abel-Jacobi map now is given by

$$(x_1, x_2) \longmapsto \left(\int_a^{x_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{dz}{\sqrt{P(z)}} \, , \int_a^{x_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{zdz}{\sqrt{P(z)}} \right)$$

Now we have to consider the two elementary symmetric functions x_1+x_2, x_1x_2. The Jacobi imversion theorem states:

The are two abelian functions φ_1, φ_2 such that

$$x_1 + x_2 = \varphi_1 \left(\int_a^{x_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{dz}{\sqrt{P(z)}}, \int_a^{x_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{zdz}{\sqrt{P(z)}} \right)$$

$$x_1 x_2 = \varphi_2 \left(\int_a^{x_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{dz}{\sqrt{P(z)}}, \int_a^{x_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{x_2} \frac{zdz}{\sqrt{P(z)}} \right)$$

Jacobi formulated this as a problem before the theorem of Riemann surfaces existed. The theory of Riemann surfaces enabled to proof this and moreover to reformulate it in a precise and very natural way. Moreover Jacobis inversion problem opened the door to the theory of complex analytic functions in several variables.

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