Lecture on Kaehler manifolds

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Chapter I. Manifolds

1. Holomorphic functions

We will frequently identify $\mathbb{C}$ and $\mathbb{R}^2$ by means of

$$z = x + \mathrm{i}y \longleftrightarrow (x, y)$$

and more generally $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ by means of

$$(z_1, \dots, z_n) \longleftrightarrow (x_1, y_1, \dots, x_n, y_n).$$

A $\mathbb{C}$-linear endomorphism of $\mathbb{C}^n$ is given by a complex $n\times n$-matrix in the usual way. The same linear map can be considered as $\mathbb{R}$-linear and then is given by a real $2n \times 2n$-matrix $\tilde{A}$. This matrix is obtained from A if one replaces each entry a by

$$\tilde{a} = \begin{pmatrix} \operatorname{Re} a & -\operatorname{Im} a \\ \operatorname{Im} a & \operatorname{Re} a \end{pmatrix}.$$

One has

$$\det \tilde{A} = |\det A|^2 \qquad (\geq 0).$$

Let V be a finite dimensional real vector space. Using an isomorphism $V \cong \mathbb{R}^n$, one equips V with a structure as topological space (i.e. the notion of an open subset with the standard derived notions as continuity, convergence, compactness, … is explained). This structure is of course independent of the choice of such an isomorphism. Similarly the ring $\mathcal{C}^\infty(U)$ of differentiable functions on an open subset is explained. For our purpose it is convenient to take these functions complex valued, $f : U \to \mathbb{C}$. This is not very essential, because by definition a complex valued function is differentiable if and only if its real and imaginary part is differentiable. In rare cases we have to consider only real functions. In such a case we use notations $\mathcal{C}^\infty(U, \mathbb{C})$ resp. $\mathcal{C}^\infty(U, \mathbb{R})$ for the complex- resp. real valued differentiable functions. A function $f \in \mathcal{C}^\infty(U)$ is called *real analytic,* if every point $a \in U$ admits a neighborhood $U(a) = U$, in which f can be expanded into a power series (which is in fact the Taylor series of f around a). This implies two facts: The Taylor series converges in $U(a)$ and the function which is represented by the Taylor series in a equals f. We denote the ring of all real analytic functions by

$$\mathcal{C}^\omega(U) \subset \mathcal{C}^\infty(U).$$

Now we are interested in the case that V is a finite dimensional *complex* vector space. We can consider V also as real vector space, hence open subsets U and the rings $\mathcal{C}^\infty(U)$ and $\mathcal{C}^\omega(U)$ are explained. Assume for the moment $V = \mathbb{C}^n$. Then we can define the Wirtinger operators

$$\frac{\partial}{\partial z_\nu},\ \frac{\partial}{\partial \bar z_\nu} : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

by

$$\frac{\partial f}{\partial z_\nu} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_\nu} - \mathrm{i}\frac{\partial f}{\partial y_\nu}\Big), \qquad \frac{\partial f}{\partial \bar z_\nu} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_\nu} + \mathrm{i}\frac{\partial f}{\partial y_\nu}\Big).$$

1.1 Definition. *A differentiable function $f \in \mathcal{C}^\infty(U)$ on an open subset of $\mathbb{C}^n$ is called **holomorphic** or **complex analytic**, if* DhF

$$\frac{\partial f}{\partial \bar z_\nu} = 0 \qquad (1 \le \nu \le n).$$

In the case $n = 1$ one writes d/dz instead of $\partial/\partial z$ (similarly with $\bar z$ instead of z). The Wirtinger operators satisfy the usual product law. Hence it is easy to apply them to polynomial expressions in z_ν and $\bar z_\nu$. We write down the rules in the case $n = 1$, the generalizations to arbitrary n are quite obvious:

$$\frac{dz^m}{dz} = mz^{m-1}, \qquad \frac{dz^m}{d\bar z} = 0.$$

This shows that a polynomial P in the variables $z_\nu, \bar z_\nu$ is a polynomial in the variables z_ν alone if $\partial P/\partial \bar z_\nu = 0$. There is a non-trivial generalization of this fact which rests on Cauchy's integral formula:

A holomorphic function $f : U \to \mathbb{C}$ admits in a suitable open neighborhood $U(a) \subset U$ of an arbitrary point a an expansion into an (absolutely convergent) power series

$$f(z) = \sum a_{\nu_1,\dots,\nu_n}(z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{z_n}.$$

As a consequence complex analytic functions are also real analytic.

It is clear that constant functions are holomorphic and that the set

$$\mathcal{O}(U) \subset \mathcal{C}^\infty(U)$$

of all holomorphic functions is a subring of the ring of all differentiable functions. Moreover the notion of holomorphic function can also be defined for abstract vector spaces V using an isomorphism $V \cong \mathbb{C}^n$.

2. Geometric spaces

We denote by $\mathcal{C}(X)$ the ring of all complex valued continuous functions on a topological space X.

2.1 Definition. *A geometric structure $\mathcal{O}$ on a topological space is a collection of subrings $\mathcal{O}(U) \subset \mathcal{C}(U)$, where U runs through all open subsets, such that the following conditions are satisfied:* DgR

1. *The constant functions are in $\mathcal{O}(U)$.*
2. *If $V \subset U$ are open sets then*

$$f \in \mathcal{O}(U) \Longrightarrow f|V \in \mathcal{O}(V).$$

3. *Let $(U_i)_{i \in I}$ be a system of open subsets and $f_i \in \mathcal{O}(U_i)$ such that*

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j \quad \textit{for all} \quad (i,j),$$

then there exist a $f \in \mathcal{O}(U)$ where $U = \bigcup_{i \in I} U_i$ with the property

$$f|U_i = f_i \quad \textit{for all} \quad i.$$

We call the functions of $\mathcal{O}(U)$ the distinguished functions. Conditions two and three mean that to be distinguished is a local property. Our main example at the moment is $X = \mathbb{C}^n$, where the distinguished functions are the holomorphic functions.

A **geometric space** is a pair $(X, \mathcal{O})$ consisting of a topological space and a geometric structure.

2.2 Definition. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is a continuous map $f : X \to Y$ with the following additional property. If $V \subset Y$ is open and $g \in \mathcal{O}_Y(V)$ then $g \circ f$ is contained in $\mathcal{O}_X(f^{-1}(V))$.* DmG

Quite trivial facts are:

The composition of two morphisms is a morphism.
The identical map $(X, \mathcal{O}) \to (X, \mathcal{O})$ is a morphism.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is called an isomorphism if f is topological and if $f^{-1} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is also a morphism. This means that the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(f(U))$ are naturally isomorphic.

Let $U \subset X$ be an open subset of a geometric space $(X, \mathcal{O})$. We can define the restricted geometric structure $\mathcal{O}|U$ by

$$\mathcal{O}|U(V) := \mathcal{O}(V) \qquad (V \subset U \text{ open}).$$

It is clear that the natural embedding $i : (U, \mathcal{O}_X|U) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism and moreover that a map $f : Y \to U$ from a geometric space $(Y, \mathcal{O}_Y)$ into U is a morphism if and only if $i \circ f$ is a morphism.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called an **open embedding**, if it is the composition of an isomorphism $(X, \mathcal{O}_X) \to (U, \mathcal{O}_Y|U)$, $U \subset Y$ open, and the natural injection.

3. Complex analytic manifolds

From now on all topological spaces are assumed to be reasonable in the following sense:

1. They are seperated, i.e. two different points admit disjoint neighborhoods.
2. They have countable basis of the topology, i.e. there exists a countable system of open subsets such that every open subset can be written as union of subsets from this system.

Usually it is clear in our examples that both properties are satisfied and we will omit the verification of these properties. The basic example of course is $\mathbb{R}^n$. As countable system can be taken the system of all balls with rational midpoints and rational radius.

3.1 Definition. *A* ***complex analytic manifold*** *is a geometric space $(X, \mathcal{O}_X)$, such that for each $a \in X$ there exists an open neighbourhood $a \in U \subset X$ and an open subset $V \subset \mathbb{C}^n$ for suitable n such that the geometric spaces $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_{\mathbb{C}^n}|V)$ are isomorphic. Here $\mathcal{O}_{\mathbb{C}^n}$ means the geometric structure on $\mathbb{C}^n$ given by the holomorphic functions. A morphism between complex analytic manifolds is also called a* ***holomorphic map*** *or complex analytic map.* DcM

It is clear that $\mathbb{C}^n$ and more general every finite dimensional complex vector space can be considered as a complex analytic manifold. The correctness of the definition of a holomorphic map shows

3.2 Proposition. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ open subset. For a map $f : U \to V$ the following two conditions are equivalent:* Bkd

1. *The map $f : (U, \mathcal{O}_{\mathbb{C}^n}|U) \to (V, \mathcal{O}_{\mathbb{C}^m}|V)$ is a holomorphic map of complex analytic manifolds.*
2. *The components of $f = (f_1, \dots, f_m)$ are holomorphic functions (contained in $\mathcal{O}_{\mathbb{C}^n}(U)$).*
3. *The map f is differentiable in the real sense and the Jacobian map (defined by the real derivatives)*

$$J(f, a) : \mathbb{C}^n \longrightarrow \mathbb{C}^m, \qquad (a \in U)$$

is not only $\mathbb{R}$*-linear but* $\mathbb{C}$*-linear.*

Additional Remark *The Jacobian map* $J(f,a)$ *can be defined as usual in real analysis usual by means of a real* $2n \times 2m$ *matrix* $J_{\mathbb{R}}(f,a)$*, which is obtained form the real partial derivatives of* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *(using the identifications* $\mathbb{C}^n = \mathbb{R}^{2n}$ *and* $\mathbb{C}^m = \mathbb{R}^{2m}$*). It also can be described by means of a complex* $n \times m$*-matrix, namely the* ***complex Jacobian***

$$J_{\mathbb{C}}(f,a) := \begin{pmatrix} \partial f_1/\partial z_1 & \dots & \partial f_1/\partial z_n \\ \vdots & & \vdots \\ \partial f_m/\partial z_1 & \dots & \partial f_m/\partial z_n \end{pmatrix}$$

The formula for connection between the two matrices $J_{\mathbb{R}}(f,a) = J_{\mathbb{C}}\widetilde{(f,a)}$ *has been explained in section 1. Recall that* $\det J_{\mathbb{R}}(f,a) = |\det J_{\mathbb{C}}(f,a)|^2$*.*

Charts

A chart on a complex analytic manifold $X = (X, \mathcal{O}_X)$ is an isomorphism

$$\varphi : (U_\varphi, \mathcal{O}_X|U) \xrightarrow{\sim} (V_\varphi, \mathcal{O}_{\mathbb{C}^n}(V_\varphi)),$$

where $U_\varphi \subset X$ and $V_\varphi \subset \mathbb{C}^n$ are open subsets. The set of all charts satisfies the following two conditions:

a) Every point of X is contained in the domain of definition of a chart.
b) If φ, ψ are two charts, then the transformation map

$$\psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \longrightarrow \psi(U_\varphi \cap U_\psi)$$

is biholomorphic.

If conversely on a topological space a set of topological maps $\varphi : U_\varphi \to V_\varphi$, $U_\varphi \subset X$ and $V_\varphi \subset \mathbb{C}^n$ open with the properties a) and b) is given, then there exists a unique structure $\mathcal{O}_X$ as complex analytic manifold, such that this set belongs to the set of charts. It should be clear how $\mathcal{O}_X$ has to be defined and that the verification is easy.

Cartesian product

Let $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ be two complex analytic manifolds. We equip $X \times X'$ with the product topology. We want to define a geometric structure. We have to define distinguished functions on open subsets of $X \times X'$. Because of the locality of the definition 2.1 it is sufficient to explain what a distinguished function on an open subset of the form $U \times U'$ with open $U \subset X$ and $U' \subset X'$ and we may assume that U and U' are domains of definitions of charts $U \to V$,

$U' \to V'$. Here $V \subset \mathbb{C}^n$ and $V' \subset \mathbb{C}^{n'}$ are open subsets. We can consider $V \times V'$ as an open subset of $\mathbb{C}^{n+n'}$. It is clear now that $\mathcal{O}_{X\times X'}(U \times U')$ has to be defined as the set of all functions which correspond to holomorphic functions on $V \times V'$. It should be clear that $(X \times X', \mathcal{O}_{X\times X'})$ is a complex analytic manifold and that the two projections $p : X \times X' \to X$ and $q : X \times X' \to X'$ are holomorphic. Moreover a map of a third complex analytic manifold Y into $X \times X'$ is holomorphic if and only if the compositions with the two projections are holomorphic.

Smooth subsets

We already defined open subspaces of geometric spaces. There is a generalization of the construction $\mathcal{O}|Y$, which equips an arbitrary subset of geometric space with a geometric structure. First one equips Y with the induced topology. Open subsets are intersections of open subsets of X with Y. Let $V \subset Y$ be an open subset. A function $f : V \to \mathbb{C}$ is called distinguished, if every point $a \in V$ admits an open neighborhood $U(a) \subset X$ (open in X), such that there exists a function $f_a \in \mathcal{O}_X(U(a))$ with the property

$$f(x) = f_a(x) \quad \text{for all} \quad x \in V \cap U(a).$$

It should be clear that this defines a geometric structure $\mathcal{O}_X|Y$ which generalizes the definition for open $Y \subset X$.

Let $(X, \mathcal{O}_X)$ be a complex analytic manifold. We are interested in conditions under which $(Y, \mathcal{O}_X|Y)$ is a complex analytic manifold too. The following little observation gives a solution: Consider $\mathbb{C}^m$ as subset of $\mathbb{C}^n$ for $m \leq n$ by means of the embedding $z \mapsto (z, 0, \dots, 0)$. let f be a holomorphic function on an open subset $V \subset \mathbb{C}^m$. Then there exists an open subset $U \subset \mathbb{C}^n$ and a holomorphic function $F : U \to \mathbb{C}$ with $F|V = f$. One can take for example $U = V \times \mathbb{C}^{n-m}$ and $F(z, w) = f(z)$. This trivial observation implies $\mathcal{O}_{\mathbb{C}^n}|\mathbb{C}^m = \mathcal{O}_{\mathbb{C}^m}$.

3.3 Definition. *A subset Y of a complex analytic manifold is called **smooth** (in the complex analytic sense), if for every point $a \in Y$ there exists a chart $\varphi : U \to V \subset \mathbb{C}^n$ such that $\varphi(U \cap Y)$ is the intersection of V with a linear subspace of $\mathbb{C}^n$.* DgA

It is clear that $(Y, \mathcal{O}_X|Y)$ is a complex analytic manifold if Y is a smooth subset of an analytic manifold $X = (X, \mathcal{O}_X)$. Hence we call $Y = (Y, \mathcal{O}_X|Y)$ a **complex analytic submanifold**. A map from a complex analytic manifold Z into Y is complex analytic if and only if the composition with the natural injection is complex analytic onto X. Especially the natural inclusion $Y \to X$ is complex analytic.

4. Differentiable and real analytic manifolds

One can replace $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ by $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ (geometric structure defined by differentiable functions) or $(\mathbb{R}^n, \mathcal{C}^\omega_{\mathbb{R}^n})$ (geometric structure defined by real analytic functions). The same definition as 3.1 leads to the notion of differentiable manifold or real analytic manifold and to the notion of differentiable map or real analytic map. We skip details.

Let $(X, \mathcal{O}_X)$ be a complex analytic manifold. Using charts we can define on X also an underlying structure $(X, \mathcal{C}^\omega_X)$ as real analytic manifold and every real analytic manifold has an underlying structure $(X, \mathcal{C}^\infty_X)$ as differentiable manifold. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a holomorphic map of analytic manifolds, then it is also a real analytic map of the underlying real analytic manifolds und similarly real analytic maps can be considered as differentiable maps of underlying differentiable manifolds.

There is also the notion of chart in the world of real analytic or differentiable manifolds. If $(X, \mathcal{O}_X)$ is a a complex analytic manifold we sometimes will call its charts holomorphic charts to distinguish them from the charts of the underlying real analytic manifold, which we hence denote real analytic charts. Similarly we use the notion "differentiable chart". Of course every holomorphic chart is real analytic and every real analytic chart is differentiable.

In the same way as in the complex analytic case one defines the cartesian product of real analytic manifolds and differentiable manifolds. Finally we mention that the construction of cartesian product is compatible with the construction of the underlying structure explained above.

There is also the notion of a smooth subset in the real analytic and differentiable worlds and hence the notion of a real analytic submanifold or differentiable submanifold. A complex analytic submanifold of a complex analytic manifold can be considered of course also as a real analytic submanifold and a real analytic submanifold of a real analytic manifold can be considered as differentiable manifold.

Dimension

Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart on a differentiable manifold. The number $\dim_a X = n$ is for all $a \in U$ independent of the choice of the chart. The dimension is a locally constant function on X. The manifold X is called pure dimensional if $\dim_a X$ is constant. Connected manifolds are pure dimensional. Let X be a complex analytic manifold. Then the dimension of X considered as (real) differentiable manifold is even, $2n$. We call n the *complex dimension* of n. A pure dimensional complex analytic manifold of (complex) dimension 1 is sometimes called a complex curve or a Riemannian surface.

Submanifolds defined by equations

Let $f : D \to \mathbb{R}^m$ a differentiable map of ein open subset $D \subset \mathbb{R}^n$. We consider the zero set

$$X = \{\, x \in X; \quad f(x) = 0 \,\}.$$

4.1 Proposition. *Assume that the Jacobian map $J(f,a) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective for all $a \in X$. (i.e. that the rank of the Jacobian matrix is m). Then* GbG

$$X = \{\, x \in X; \quad f(x) = 0 \,\}$$

is smooth and its dimension in all points is $n - m$. The analogue statement is true in the real analytic and complex analytic worlds.

5. Examples of manifolds

We give some examples of complex analytic manifolds. Some constructions are based on the following general construction for geometric spaces. Let $(X, \mathcal{O}_X)$ be a geometric space and let Γ be a group of automorphisms of $(X, \mathcal{O}_X)$. (An automorphism is an isomorphism onto itself.) We recall that Γ induces an equivalence relation on X. Two points a, b are called equivalent if there exists a $\gamma \in \Gamma$ with $\gamma(a) = b$. We denote by $Y := X/\Gamma$ the set of equivalence classes. There is a natural projection $\pi : X \to Y$. We equip Y with the quotient topology. This means that a subset $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open in X. Then $\pi : X \to Y$ is continuous (which means that inverse images of open sets of Y are open) and also open (which means that images of open sets of X are open). We equip Y with a geometric structure. A function $h : V \to \mathbb{C}$ is called distinguished if and only if $h \circ \pi : \pi^{-1}(V) \to \mathbb{C}$ is distinguished. It is easy to see that this is a geometric structure $\mathcal{O}_Y$. The geometric space obtained in this way is called the quotient space. A good way to look at this structure is as follows: Consider for open $V \subset Y$ the natural map

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\pi^{-1}(V)).$$

It is clear that this map is injective and that its image consists of all Γ-invariant elements, i.e. of functions $f \in \mathcal{O}_X(\pi^{-1}(V))$ with the property

$$f(\gamma(x)) = f(x) \qquad (\pi(x) \in V,\ \gamma \in \Gamma).$$

In a self explaining notation this means

$$\mathcal{O}_Y(V) \cong \mathcal{O}_X(\pi^{-1}(V))^\Gamma.$$

(If a group arises as upper index this usually means “taking invariants”.)

Even when X is a seperated, the quotient Y needs not to be separated. The condition that Y is seperated means that two points $x_1, x_2 \in X$ with different image points in Y admit neighborhoods $U_1, U_1 \subset X$ sucht that no point of U_1 is equivalent to some point of U_2. We assume now that X and Y both are seperated.

By definition the group Γ **acts freely** on X, if the map $\pi : X \to X/\Gamma$ is locally topological. This is equivalent to the following fact: Every point $a \in X$ contains an open neighborhood U such that two different points of U are inequivalent mod Γ. Then $V = \pi(U)$ is open in Y and the restriction of π defines a topological map from U onto V. We assumed that X carries a geometric structure such that Γ respects this structure. Then it is clear that the map $(U, \mathcal{O}_X|U) \to (V, \mathcal{O}_{X/\Gamma}|V)$ is an isomorphism of geometric spaces. We obtain:

5.1 Remark. *Let X be a complex analytic manifold and Γ a group of biholomorphic mappings of X onto itself, which acts freely on X. Then X/Γ carries also a structure as complex analytic manifold. A map $X/\Gamma \to Y$ to another complex analytic manifold is holomorphic if and only if its composition with the natural projection $X \to X/\Gamma$ is holomorphic.* UcQ

The same remark also holds in the worlds of real analytic or differentiable manifolds.

We treat a very simple example. A subgroup $L \subset V$ of a real vector space is called a lattice if there exists a basis $e_1, \dots, e_n$ with the property

$$L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n.$$

This means that there exists an isomorphism $V \to \mathbb{R}^n$ such that the image of L is $\mathbb{Z}^n$. This shows that L is an additive group which is isomorphic to $\mathbb{Z}^n$ Every $a \in L$ induces a map

$$V \longrightarrow V, \quad x \longmapsto x + a.$$

The group of mappings obtained in this way is isomorphic to L. We hence simply write V/L for the quotient of V by this group. Two elements $x, y \in V$ have the same image in V/L if and only if $x - y \in L$. Hence V/L coincides with the construction of the factor group of an abelian group by a subgroup.

A real torus by definition is a real analytic manifold which is isomorphic as real analytic manifold to V/L where V is a finite dimensional vector space and L a lattice in V. Every real torus is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. A **complex torus** is a complex analytic variety X which is biholomorphic equivalent to V/L where V is a finite dimensional complex vector space and $L \subset V$ a lattice (of the underlying real vector space).

Two real tori of the same dimension are isomorphic as real analytic manifolds. But two complex tori of the same dimension are usually not biholomorphic equivalent.

The projective space

We consider a complex vector space of finite dimension. We denote by $P(V)$ the set of all lines (one dimensional complex sub-vector spaces). There is a natural map

$$V - \{0\} \longrightarrow P(V), \quad a \longmapsto \mathbb{C}a.$$

We denote by $[v] := \mathbb{C}v$ the image of a $v \in V - \{0\}$. Two points a, b define the same image, $[a] = [b]$ if there exists a complex number $\alpha \neq 0$ with the property $b = \alpha a$. We consider the group of all transformations

$$V - \{0\} \longrightarrow V - \{0\}, \quad z \longmapsto \alpha z,$$

where $\alpha \neq 0$ is a complex number. This group of transformations is isomorphic to the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. We may write this as

$$P(V) = (V - \{0\})/\mathbb{C}^*.$$

This action is of course not free. Nevertheless we can consider the quotient structure $\mathcal{O}_{P(V)}$.

5.2 Remark. *The complex projective space is a compact complex analytic manifold of dimension* $\dim(V) - 1$. Cps

We describe the structure of the projective space more closely in the case $V = \mathbb{C}^{n+1}$. Let $U_i \subset P(V)$ the subset of all points represented by a $(z_0, \ldots, z_n)$ with $z_i \neq 0$. We will show that $(U_i, \mathcal{O}_{P(V)}|U_i)$ and $(\mathbb{C}^n, \mathcal{O}_\mathbb{C})$ are isomorphic. For simplicity of notation we assume $i = 0$. Then the map

$$\mathbb{C}^n \longrightarrow U_0, \quad z \longmapsto [1, z_1, \ldots, z_n]$$

has the desired property. The inverse map is given by

$$[z_0, \ldots, z_n] \longmapsto \Big(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\Big).$$

The compactness of $P(V)$ follows from the map that the restriction of the projection π to the subset $\|v\| = 1$ is surjective. Here $\|\cdot\|$ denotes an arbitrary norm. This completes the proof of 5.2.

We mention two additional obvious facts:

a) *The affine pieces U_i are open and dense in $P^n(\mathbb{C}) := P(\mathbb{C}^{n+1})$.*
b) *The complements $P^n(\mathbb{C}) - U_i$ are smooth. There is a natural biholomorphic map*

$$P^n(\mathbb{C}) - U_i \cong P^{n-1}(\mathbb{C}).$$

A special case is $P^1(\mathbb{C})$. It is the union of an affine part $\mathbb{C}$ and one additional point.

$$P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}.$$

Sometimes $P^1(\mathbb{C})$ is called the Riemannian sphere.

Algebraic varieties

Let P be a homogenous polynomial in $n+1$ variables $z_0, \dots, z_n$. Homogenous of degree k means $P(tz) = t^k P(z)$. When P vanishes on a point $a \in \mathbb{C}^{n+1}$ it vanishes on the whole $\mathbb{C}a$. Hence we can consider the set of zeros of P on the projective space $P^n(\mathbb{C}) := P(\mathbb{C}^{n+1})$. By definition a projective algebraic variety is a subset of $P^n(\mathbb{C})$ which can be defined as the set of common zeros of a finite system of homogenous polynomials

$$X = \left\{ x \in P^n(\mathbb{C}); \quad P_1(z) = \ldots = P_m(z) = 0 \right\}.$$

It may happen that X is a (complex analytic) smooth submanifold. There is a famous **theorem of Chow** which we will not use in these notes but which is behind the scenes:

Every closed complex analytic (smooth) submanifold of $P^n(\mathbb{C})$ is algebraic.

We give an example. Consider the polynomial

$$P(t, z, w) := t^4 w^2 - 4t^3 z^3 - g_2 t^5 z - g_3 t^6$$

We assume that g_2, g_3 are arbitrary complex numbers such that $g_2^3 \neq 27 g_3^2$. One can check that this means nothing else but that the cubic polynomial $4z^3 - g_2 z - g_3$ has no multiple zero. It can be checked that the set of zeros $X(g_2, g_3)$ of P is smooth in $P^2(\mathbb{C})$. It is a so-called elliptic curve. From the theory of elliptic functions follows that $X(g_2, g_3)$ is biholomorphic equivalent to a complex torus $\mathbb{C}/L$ and conversely that every complex torus is biholomorphic to such an elliptic curve.

Chapter II. Vector bundles

1. The notion of a vector bundles

The notion of a vector bundle is in principle a topological notion:

1.1 Definition. *A vector bundle on a topological space X is a topological space E and a surjective continuous mapping $\pi : E \to X$ such that every fibre $E_x := \pi^{-1}(x)$ $(x \in X)$ carries a structure as finite dimensional vector space and such that the following conditions holds:* DVb

Every point $a \in X$ admits an open neighbourhood $U(a)$ and a topological map

$$\pi^{-1}(U) \longrightarrow U \times W$$

for a suitable finite dimensional vector space W with the following properties:

a) *The following diagram commutes:*

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times W \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

(The first vertical arrow is given by π, the second is a natural projection.)

b) *The induced mappings*

$$E_x \longrightarrow W \qquad (x \in U)$$

are vector space isomorphisms.

We left it open in the definition whether we work with real or complex vector spaces. Both is possible. When we want to specify which sub-field is taken, then we talk about a real or a complex vector field. We are mainly interested in complex vector fields.

For the construction of vector bundles it is useful that the assumptions in the definition of a vector bundle can be weakened.

1.2 Remark. *Let X be a topological space and E a set together with a surjective map $\pi : E \to X$ such that every fibre carries a structure as vector space. Assume that there is given an open covering $X = \bigcup_\alpha U_\alpha$ and a system of bijective mappings* RDv

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times W_\alpha,$$

where W_α is a finite dimensional vector space, which satisfy the conditions a) and b) in 1.1and such that the following condition holds:

If α, β are two indices of this system then

$$h_\beta \circ h_\alpha^{-1} : U_\alpha \cap U_\beta \times W_\beta \longrightarrow U_\alpha \cap U_\beta \times W_\alpha$$

is a topological map.

Then E carries a unique topology that all the h are topological maps. Especially $E \to X$ is a vector bundle.

It is clear how to define the topology: A set of E has to be defined open if and only if the intersection with any $\pi^{-1}(U_\alpha)$ is mapped by h to an open set of $U_\alpha \times W_\alpha$ (which of course is equipped with the product topology.

1.3 Definition. *Let X be a complex analytic manifold. A* ***holomorphic vector bundle*** *on X is a complex analytic space E and a surjective holomorphic map $\pi : E \to X$ such that every fibre E_x carries a structure as* ***complex vector space*** *with the same property as in 1.1 and the additional property, that the local trivializations $\pi^{-1}(U) \to U \times W$ can be chosen as biholomorphic maps.* DaV

In a similar way one defines the notion of differentiable vector bundle on a differentiable manifold or real analytic vector bundle on a real analytic manifold. In these two cases one can consider ether real or complex vector bundles.

There are obvious variants of 1.2 in the three cases.

Transition functions

Let $\pi : E \to X$ be a vector bundle. When X is connected, all the spaces E_x must have the same dimension because the function $x \mapsto \dim E_x$ is locally constant. Hence we can take for all local trivializations the same vector space W, for example $\mathbb{R}^n$ or $\mathbb{C}^n$. Let's assume that we have one W. Recall that for two local trivializations $h : \pi^{-1}(U) \to U \times W$ and $h' : \pi^{-1}(U') \to U' \times W$ we obtain a map $g : U \cap U' \times U \cap U' \times W$. This map can be considered also as map

$$g : U \cap U' \longrightarrow \mathrm{GL}(W).$$

This map is called the transition map. A bundle atlas is a system of local trivializations $h : \pi^{-1}(U) \to U \times W$ with the properties a) and b) in 1.1 and such that these open subset U cover X. By definition every vector bundle admits a bundle atlas. The transition maps satisfy an obvious relation.

1.4 Remark. *Assume that three local trivalizations* (U_α, h_α), (U_β, h_β), (U_γ, h_γ) *of a vector bundle* $E \to X$ *are given. The transition functions* Rlt

$$g_{\alpha,\beta} = h_\alpha \circ h_{\beta^{-1}} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(W)$$

(similarly $g_{\beta,\gamma}$, $g_{\gamma,\alpha}$*) satisfy the relation*

$$g_{\alpha,\beta} \circ g_{\beta,\gamma} \circ g_{\gamma,\alpha} = \mathrm{id}_W \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

It is possible to reconstruct the vector bundle from an atlas of transition functions:

1.5 Remark. *Let* X *be a topological space and* W *a finite dimensional vector space. Assume that an open covering* $X = \bigcup U_\alpha$ *by open subsets* U *together with a continous map* $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(W)$ *for each pair* $(\alpha, \beta))$ *of this system is given. Then there exists a vector bundle* $E \to X$ *such that all* $g_{\alpha,\beta}$ *are transition functions for this vector bundle. This vector bundle is unique up to isomorphism.* CvT

Sketch of proof. Consider the *disjoint union*

$$\tilde{E} = \bigcup_\alpha U_\alpha \times W.$$

One defines an equivalence relation: Two pairs $(a, u) \in U_\alpha \times W$ and $(b, v) \in U_\beta \times W$ are called equivalent, $(a, u) \sim (b, v)$, if and only if $a = b$ and $g_{\alpha,\beta} u = v$. Then we define $E = \tilde{E}/\sim$ as the quotient (set of all equivalence classes). We equip E with the quotient topology $E \to X$. It is clear, how the fibres of this map get their vector space structure. □

We mention that the transition map for a holomorphic vector bundle are holomorphic maps and that there is also a holomorphic version of 1.5 and the same is true for the real analytic and differentiable cases.

2. Examples of vector bundles

The easiest example is the trivial vector bundle $X \times W \to X$. Every vector bundle $\pi : E \to X$ is locally trivial by definition, i.e. X can be covered by open subsets U such that the bundle $\pi : \pi^{-1}(U) \to U$ is trivial. We start with a non-trivial example which can be described by means of transition function and give afterwards an even better coordinate invariant description. Let X be a pure dimensional differentiable manifold. The index set which we consider is the set of all (differentiable) charts $\alpha : U_\alpha \to V_\alpha$. For two charts α, β we have the transformation map $\gamma = \beta \circ \alpha^{-1}$. For every $a \in U_\alpha \cap U_\beta$ we can consider the Jacobian

$$g_{\alpha,\beta} := J(\gamma, x) \in \mathrm{GL}(\mathbb{R}^n) \qquad (x = \alpha(a)).$$

The chain rule confirms that this satisfies the assumptions of transition functions (1.5). The corresponding differentiable vector bundle is called the **tangent bundle**. We have defined it as a real vector bundle. In the same way we define the analytic tangent bundle for real analytic manifolds. Usually this tangent bundles are written as TX. The fibre over a point is a real vector space of dimension $\dim X$.

There is a variant in the case of complex analytic manifolds. Here were consider only holomorphic charts. The transformation $\gamma = \beta \circ \alpha^{-1}$ is is biholomorphic and we can consider its complex Jacobian:

$$g_{\alpha,\beta}(a) = J_{\mathbb{C}}(\gamma, z) \in \mathrm{GL}(\mathbb{C}^n) \qquad (z = \gamma(a)).$$

This leads now to a holomorphic vector bundle which we call the holomorphic tangent bundle of X. The holomorphic tangent bundle is a complex vector bundle. But every complex vector bundle can also be considered as a real vector bundle in the same manner as a complex vector space can be considered as a real vector space. We also have mentioned that the complex Jacobian $J_{\mathbb{C}}(\gamma, z) \in \mathrm{GL}(\mathbb{C}^n)$ equals the real Jacobian $J_{\mathbb{R}}(\gamma, z) \in \mathrm{GL}(\mathbb{R}^{2n})$ when we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and consider $\mathrm{GL}(\mathbb{C}^n)$ as a subgroup of $\mathrm{GL}(\mathbb{R}^{2n})$ This shows:

Let X be a complex analytic manifold. The real vector bundle which underlies the holomorphic tangent bundle is the real analytic tangent bundle of the real analytic manifold which underlies X.

We give now a coordinate invariant description of the tangent bundle: For this purpose we introduce the stalk $\mathcal{O}_{X,x}$ of a geometric structure in a point $x \in X$. For this we define the set of all pairs (U, f) where U is an open neighbourhood and $f \in \mathcal{O}_X(U)$. This can be considered as the disjoint union of all $\mathcal{O}_X(U)$. We introduce an equivalence relation: Two pairs (U, f), (V, g) are called equivalent if there exists an open neighborhood $a \in W \subset U \cap V$ with the property $f|W = g|W$. This is an equivalence relation. The equivalence classes are so-called germs in a. We denote the equivalence class represented by an (U, f) by

$$f_a = [U, f]_a.$$

On can add germs

$$[U, f]_a + [V, g]_a := [U \cap V, f + g]_a.$$

It is clear that this is definition is independent of the choice of the representatives. In the same way on defines the product of two germs and the product of germ with a scalar. We see that the set $\mathcal{O}_{X,x}$ is equipped with a structure as commutative ring and also with a structure as (complex) vector space.

By an associative algebra A over a field K we understand an associative ring with unity 1_A together with a ring homomorphism $\alpha : K \to A$ which sends 1_K to 1_A. It is easy to see that α is injective when $1_A \neq 0_A$. This is the case if A contains one element different from zero. So besides this trivial case we can consider K as subring of A. Hence we surpress usually α. Then A can be considered as K-vector space by means of $ka := \alpha(k)a$. There is an obvious notion of homomorphism between two K-algebra. We always assume that the unit element is mapped to the unit element.

2.1 Definition. *Let R, S be two associative algebras over a field K. A derivation is a K-linear map $D : R \to S$ with the property* DDr

$$D(ab) = aD(b) + D(a)b.$$

We denote the K-vector space of all derivations by $\mathrm{Der}(R, S)$.

From the product rule follows $D(1_A 1_A) = D(1_A) + D(1_A)$, hence $D(1_A) = 0$. From the K-linearity follows $D(k1_A) = 0$ for all $k \in K$. We are interested in the vector space $\mathrm{Der}(\mathcal{O}_{X,x}, \mathbb{C})$. (The ground field is $\mathbb{C}$). This space can be described in a different way. Let

$$\mathfrak{m}_a = \mathfrak{m}(\mathcal{O}_{X,a})$$

be the set of all germs f_a such that $f_a(a) = 0$. (The value $f_a(a) := f(a)$ is independent of the choice of a representative.) We denote by $\mathfrak{m}_a^2$ the vector space generated by all $f_a g_a$, $f_a, g_a \in \mathfrak{m}_a$. From the product rule follows that an element $D \in \mathrm{Der}(\mathcal{O}_{X,x}, \mathbb{C})$ vanishes on $\mathfrak{m}_a^2$. Hence D induces a linear form on $\mathfrak{m}_a/\mathfrak{m}_a^2$. The space of all linear forms is the dual vectors space $(\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. We have seen that there is a natural linear map

$$\mathrm{Der}(\mathcal{O}_{X,x}, \mathbb{C}) \longrightarrow (\mathfrak{m}_a/\mathfrak{m}_a^2)^*.$$

We claim that this map is an isomorphism. For this we describe the inverse map. Let L be a linear form on $\mathfrak{m}_a/\mathfrak{m}_a^2$. We define $D(f) := L(f - f(a))$ in an obvious notation. The product rule follows from $L((f - f(a))(g - g(a)) = 0$.

We now consider special cases. Assume that X is a differentiable manifold and $a \in X$. We choose a chart $\varphi : U \to V \subset \mathbb{R}^n$. Let $f_a \in \mathcal{C}^\infty_{X,a}$. We may assume that f_a is represented by a function f on U and we may consider the corresponding $g : V \to \mathbb{C}$. We can define

$$\Big[\frac{\partial}{\partial x_\nu}\Big]_a f_a := \frac{\partial g}{\partial x_\nu}\Big|_{x=a}.$$

This defines a derivation. The point is that these n Derivations build a basis of the vector space of all derivations. This follows from the following basic fact:

2.2 Remark. *Let be* Rdt

$$\mathfrak{m}_a = \mathfrak{m}(\mathcal{C}^\infty_{\mathbb{R}^n,a}).$$

The space $\mathfrak{m}_a/\mathfrak{m}_a^2$ *is* n*-dimensional. A basis is given by the (classes of) linear functions* $p_\nu(x) = x_\nu - a_\nu$.

Proof. This is a consequence of Taylor's formula. From this formula (for example in its integral version) follows that any $f_a \in \mathcal{C}^\infty_{\mathbb{R}^n,a}$ can be written as

$$f_a = f_a(a) + \sum_{\nu=1}^{n} C_\nu(x_\nu - a_\nu) + h_a,$$

where h_a is represented by a function

$$h = \sum_{\nu,\mu}(x_\nu - a_\nu)(x_\mu - a_\mu)h_{\nu,\mu},$$

where $h_{\mu\nu}$ are differentiable functions in an open neighborhood of a. Clearly $h_a \in \mathfrak{m}_a^2$. □

Assume that $\mathcal{O}_X$ has the property that with f also $\bar{f}$ is contained in $\mathcal{O}_X$. Then $\mathcal{O}_X$ is determined by all its functions, which are real valued. In this case it is sufficient to consider only real derivations $D \in \mathrm{Der}(\mathcal{O}_{X,x}, \mathbb{C})$, where D is called real if $D(\bar{f}) = \overline{D(f)}$. It is clear that 2.2 has also a real variant which leads to

2.3 Proposition. *Let* X *be a pure* n*-dimensional differentiable manifold and* $a \in X$*. We denote by* $T_aX \subset \mathrm{Der}(\mathcal{C}^\infty_{X,a}, \mathbb{C})$ *the vector space of all real derivations. This is a real vector space of dimension* n*. If* $U \to V \subset \mathbb{R}^n$ *is a chart on some open neighborhood* U *of* a*, then the derivations* $[\partial/\partial x_\nu]_a$*,* $1 \le \nu \le n$*, form a basis of* T_aX*.* TbP

We denote by E the set of all pairs (a, A) where $a \in X$ and $A \in T_aX$. There is a natural projection $pi : E \to X$ We see from 2.3 that for every chart φ we obtain a bijective map

$$\pi^{-1}(U_\varphi) \xrightarrow{\sim} U_\varphi \times \mathbb{R}^n.$$

We can imply 1.2 and obtain a structure as real vector bundle for $E \to X$. It is easy to compute the transition functions of this vector bundle. The chain rule shows that they are given by the Jacobian matrix, i.e. the transition functions are the same which we used for the construction of the tangent bundle TX. This means that we obtain a natural isomorphism $TX \cong E$.

2.4 Proposition. *The fibres of the tangent bundle* TX *can be identified with the space* $T_aX \subset \mathrm{Der}(\mathcal{C}^\infty_{X,a}, \mathbb{C})$ *of real derivations.* ITd

The same is true for the real analytic world. The complex analytic world is slightly different. For a complex analytic manifold $(X, \mathcal{O}_X)$ we consider the germs of holomorphic functions and of course all derivations. For a moment we denote them by

$$T_a^{\rm hol}X := \mathrm{Der}(\mathcal{O}_{X,a}, \mathbb{C}).$$

In the same manner as for differentiable manifolds we obtain a holomorphic tangent bundles $T^{\rm hol}X$. The transition matrices are the complex Jacobians. We have already seen that the real bundle which underlies this bundle can be identified with the real tangent bundle.

2.5 Remark. *The real vector bundle which underlies the complex bundle $T^{\rm hol}X$ is naturally isomorphic to the real tangent bundle TX. The isomorphism* RgC

$$T_aX \longrightarrow T_a^{\rm hol}X$$

is defined by restriction of derivations.

Because $T^{\rm hol}X$ is only a complex structure on the real tangent bundle there is no need for an extra notation. In future we usually will write TX instead of $T^{\rm hol}X$.

It is worth while to describe the identification

$$T_a^{\rm hol}X = T_aX$$

in local coordinates, i.e. for $X = \mathbb{C}^n$. The $\mathbb{C}$-basis of $T_a^{\rm hol}X$ is given by the partial derivatives $[\partial/\partial z_\nu]_a$. The $\mathbb{R}$-basis of T_aX is given by $[\partial/\partial x_1]_a, [\partial/\partial y_1]_a, \dots$. The isomorphism of the tangent spaces is given by

$$\sum_{i=1}^n \alpha_i \Big[\frac{\partial}{\partial x_i}\Big]_a + \sum_{i=1}^n \beta_i \Big[\frac{\partial}{\partial y_i}\Big]_a \longleftrightarrow \sum_{i=1}^n (\alpha_i + \mathrm{i}\beta_i) \Big[\frac{\partial}{\partial z_i}\Big]_a.$$

Recall again that this means that the operators on both sides have the same effect on *holomophic* functions. The tangent space $T_a^{hol}X$ is a complex vector space Hence there must be an intrinsic multiplication with i inside the real vector space T_aX This has nothing to do with the operators $\mathrm{i}[\partial/\partial x_\nu]_a, \mathrm{i}[\partial/\partial y_\nu]_a$ because this are not inside of the real tangent space. This is a good reason to denote the intrinsic multiplication with i by a new letter J:

2.6 Remark. *Let X be a complex analytic manifold, a a point of X with a holomorphic chart around a. The intrinsic multiplication with* i *inside the tangent space T_aX is* CSd

$$J\Big[\frac{\partial}{\partial x_\nu}\Big]_a = \Big[\frac{\partial}{\partial y_\nu}\Big]_a, \quad J\Big[\frac{\partial}{\partial y_\nu}\Big]_a = -\Big[\frac{\partial}{\partial x_\nu}\Big]_a.$$

The tautological bundle of the projective space

We consider the complex projective space $P^n(\mathbb{C})$ defined as the set of lines $l \subset \mathbb{C}^{n+1}$. We consider pairs (l, a) where l is a line and $a \in l$. We denote be L the set of all these pairs. There is a natural projection

$$\pi : L \longrightarrow P^n(\mathbb{C}), \quad (l, a) \longmapsto l.$$

The fibres are

$$L_l = l.$$

we consider an affine piece of $P^n(\mathbb{C})$. For simplicity of notation we take the part U_0 defined by $z_0 \neq 0$. We recall that U_0 can be identified with $\mathbb{C}^n$. A point $z \in \mathbb{C}^n$ corresponds to the line generated by $(1, z)$ There is a natural bijection

$$\mathbb{C}^n \times \mathbb{C} \xrightarrow{\sim} \pi^{-1}(U_0), \quad (z, t) \longmapsto ([1, z], t(1, z)).$$

Recall that $[1, z]$ is the line which contains $(1, z)$ and notice that $t(1, z)$ is a point of this line. Identifying $\mathbb{C}^n$ with U_0 we also get a natural bijection

$$U_0 \times \mathbb{C} \cong \pi^{-1}(U_0).$$

Applying the holomorphic variant of 1.2 we obtain a holomorphic vector bundle on $P^n(\mathbb{C})$. This bundle is called tautological.

3. Some linear algebra

We recall basic operations od linear algebra. In the following we consider finite dimensional vector spaces over an arbitrary field K.

let V, W be two vector spaces,

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_K(V, W) = \big\{ f : V \longrightarrow W; \quad f \ K\text{-linear} \big\}.$$

One has $\dim \mathrm{Hom}(V, W) = \dim V \cdot \dim W$.

A special case is the dual space

$$V^* = \mathrm{Hom}_K(V, K).$$

The dual space is contravariant, i.e. a linear map $f : V \to W$ induces an obvious linear map $f^* : W^* \to V^*$. Let $e_1, \ldots, e_n$ be a basis of V then one obtains a basis $e_1^*, \ldots, e_n^*$ of V^* by

$$e_i^*(e_j) := \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

Hence V and V^* have the same dimension. Hence they are isomorphic, but the isomorphim is not canonical, it depends on the choice of basises.

Let $V_1, \dots, V_n$, W be vector spaces. The set of multilinear mappings $V_1 \times \cdots \times V_n \to W$

$$\text{Mult}(V_1 \times \cdots \times V_n, W)$$

is a space of dimension $\dim V_1 \cdots \dim V_n \cdot \dim W$. In the case $W = K$ one talks of multilinear *forms.*

Pairings

let

$$V \times W \to K, \quad (a,b) \longmapsto \langle a,b\rangle.$$

be a bilinear form. There are induced linear maps

$$V \longrightarrow W^*, \qquad W \longrightarrow V^*.$$

For example the first on attaches to an element $a \in V$ the linear form

$$l_a(x) := \langle a,x\rangle \quad (x \in W).$$

There are two important special cases:

In the case

$$V \times V^* \longrightarrow K, \quad \langle a,l\rangle := l(a)$$

one obtains a linear map $V^* \to V^*$ which turns out to be the identity and a linear map

$$V \longrightarrow V^{**}$$

which is more interesting. Using a basis, one sees that this map is an isomorphism. (This depends heavily on our assumption that V is finite dimensional.) We see that V and V^{**} are *canonically isomorphic.*

Another interesting case is $V = W$, i.e.

$$V \times V \longrightarrow K.$$

Hence we get two maps $V \to V^*$. We are mainly interested in the case that this pairing is symmetric, then both maps agree: Following properties are equivalent:

One of the maps $V \to V^*$ is an isomorphism.
Both are isomorphims
If $e_1, \dots, e_n$ is a basis, then the so-called Gram-matrix

$$(\langle e_i, e_j\rangle)_{1\le i,j\le n}$$

is regular.

If this is the case, we call the pairing *non-degenerated.*

The tensor product

For finite dimensional vector spaces $V_1, \dots, V_n$ the tensor product can be defined as follows:

$$V_1 \otimes \ldots \otimes V_n := \mathrm{Mult}(V_1^* \times \cdots \times V_n^*, K).$$

It is important that it comes up together with a multilinear mapping

$$V_1 \times \ldots \times V_n \longrightarrow V_1 \otimes \ldots \otimes V_n, \quad (v_1, \dots, v_n) \longmapsto v_1 \otimes \cdots \otimes v_n,$$

where $v_1 \otimes \cdots \otimes v_n$ is the following multilinear form

$$v_1 \otimes \cdots \otimes v_n \; (l_1, \dots, l_n) := l_1(e_1) \cdots l_n(e_n).$$

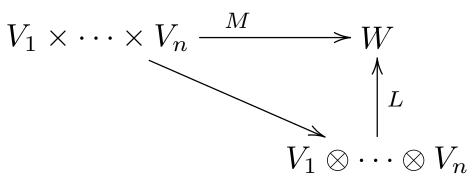
The pair satisfies the following property:
Let $e_1^{(i)}, \dots, e_{n_i}^{(i)}$ be a basis of V_i, then the elements

$$e_{\nu_1}^{(1)} \otimes \cdots \otimes e_{\nu_n}^{(n)}$$

perform a basis of $V_1 \times \ldots \times V_n$. Obviously

$$\dim \; V_1 \times \ldots \times V_n = \dim V_1 \cdots \dim V_n.$$

An easy consequence of this property is the
Universal property of the tensor product. Let $M : V_1 \times \cdots \times V_n \to W$ be a *multilinear* map into an arbitrary vector space. then there exists a unique *linear* map $L : V_1 \otimes \cdots \otimes V_n \to W$, such that the following diagram commutes:



This means

$$L(v_1 \otimes \cdots \otimes v_n) = M(v_1, \dots, v_n).$$

In this connection on should mention that the tensor product is generated by so-called pure tensors $v_1 \otimes \cdots \otimes v_n$. Hence a linear map L is known, if one knows its action on pure tensors. But the pure tensors are not linearly independent, hence on has to be careful with a definition which assigns to pure tensors something.

The proof of the universal property nevertheless is trivial. One first restricts to pure tensors, where the a_i are taken from a fixed basis o V_i. Then the definition is possible. Henceafter one uses the multilinearity of M to verify that the formula is for arbitrary a_i valid. We will often define linear maps by there action on pure tensors. One have always to have the universal property in mind to justify this. Usually we will not mention this:

It can be shown that the universal property characterizes the tensor product up to canonical isomorphism. This universal property (and not the ad hoc defnition we gave) is also the correct starting point for the definition of the tensor product if not necessarily finite dimensional vector spaces or even more general for modules over a ring. We will avoid to use this in our notes (but it would be helpful for some of the constructions).

By means of the universal property one proves easily the associativity of the tensor product: For $1 < a < n$ one has the following isomorphism

$$\begin{aligned}(V_1 \otimes \cdots \otimes V_a) \otimes (V_{a+1} \otimes \cdots \otimes V_n) &\xrightarrow{\sim} V_1 \otimes \cdots \otimes V_n,\\ (v_1 \otimes \cdots \otimes v_a) \otimes (v_{a+1} \otimes \cdots \otimes v_n) &\longmapsto v_1 \otimes \cdots \otimes v_n.\end{aligned}$$

and also the commutativity: Let σ be a permutation of the digits $1, \dots, n$. One has the isomorphism

$$V_1 \otimes \cdots \otimes V_n \xrightarrow{\sim} V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}, \quad v_1 \otimes \cdots \otimes v_n \longmapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

A interesting case is also the tensor product of d copies of the same vector space V:

$$V^{\otimes d} := \overbrace{V \otimes \cdots \otimes V}^{d}.$$

We set additionally

$$V^{\otimes 0} = K, \quad V^{\otimes 1} = V.$$

Every permutation σ of the digits $1, \dots, d$ induces an automorphism

$$V^{\otimes d} \otimes V^{\otimes d}, \quad v_1 \otimes \cdots \otimes v_d \longmapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

We denote the image of an element $T \in V^{\otimes d}$ simply by T^σ. An element T is called *alternating,* if

$$T^\sigma = \operatorname{sgn}(\sigma)(T).$$

The subspace of alternating tensors is denoted by

$$\bigwedge^d V := \{\, T \in V^{\otimes d}; \quad T \text{ alternating} \,\},$$

additionally

$$\bigwedge^0 V = K, \quad \bigwedge^1 V = V.$$

There is a natural projection

$$V^{\otimes d} \longrightarrow \bigwedge^d V, \quad T \longmapsto T^{alt} := \frac{1}{d!} \sum_{\sigma} T^{\sigma},$$

which is the identity on the alternating part. This gives a pairing

$$\bigwedge^p V \times \bigwedge^q V \longrightarrow \bigwedge^{p+q} V, \quad (A, B) \longmapsto A \wedge B := (A \otimes B)^{alt}.$$

In the cases $p = 0$ and $q = 0$ this understood as usual skalar multipliction with constants.

The spaces $\bigwedge^\bullet V$ with the wedge product $\wedge$ perform the so-called Grassmann algebra.

We conclude this excursion into linear algebra by a trivial remark which again follows from the universal property: Let $L_i : V_i \to W_i$ be linear maps $(1 \leq i \leq n)$. Then there is an induced linear map

$$L : V_1 \otimes \cdots \otimes V_n \longrightarrow W_1 \otimes \cdots \otimes W_n, \quad v_1 \otimes \cdots \otimes v_n \longmapsto L_1(w_1) \otimes \cdots \otimes L_n(v_n).$$

Similarly a linear map $V \to W$ induces linear maps

$$\bigwedge^d V \longrightarrow \bigwedge^d W.$$

They are compatible with the wedge product.

There is an interesting application: Let $n = \dim V$. Then $\bigwedge^d V = 0$ for $d > n$. Moreover

$$\dim \bigwedge^d V = \binom{n}{d}.$$

especially

$$\dim \bigwedge^n V = 1 \qquad (n = \dim V).$$

An endomorphism $A : V \to V$ induces an endomorphism of the one-dimensional space $\bigwedge^n V$. Such an endomorhism is multiplication with a scalar α. Actually

$$\det A = \alpha.$$

4. Bundle maps

In the previous section we described certain constructions for vector spaces as the Hom-spaces, Mult-spaces, dual space, the tensor product and the Grassmann-products. All these constructions can be performed also for vector bundles. We explain this in one example, from which everything should be clear. We take the case of the dual bundle: Let $\pi : E \to X$ be a real vector bundle over the topological space X. We consider for each $a \in X$ the dual space E_a^* of the fibre E_a and denote by

$$E^* := \dot{\bigcup} E_a^*$$

there disjoint union and by $\pi^*; E^* \to X$ the natural projection. For simplicity we assume that the E_a^* are disjoint in advance. (Otherwise one had to consider pairs (a, A) with $a \in X$ and $A \in E_a$.) as in 1.1 we consider local trivializations $\pi^{-1}(U) \to U \times \mathbb{R}^n$. For each $a \in U$ we obtain a basis $e_1, \dots, e_n$ from E_a which maps to the standard basis of $\mathbb{R}^n$. Consider the dual basis $e_1^*, \dots, e_n^*$. This gives an isomorphism $E_a^* \to \mathbb{R}^n$ and hence a bijection

$$(\pi^*)^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n.$$

From 1.2 we see that there is a unique bundle structure on E^* such that these maps are local trivializations.

4.1 Definition. *Let $E, F \to X$ be two vector bundles. A continuous map $f : E \to F$ is called a **bundle map**, if the diagram* DLv

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes and if the induced maps of the fibres $f_a : E_a \to F_a$ are linear.

This definition can be made for real and complex vector bundles and there are also obvious variants in the holomorphic, real analytic or differentiable worlds.

It is clear the the composition of bundle mappings is a bundle map and that the identity is a bundle map. As a consequence the notion of isomorphism of vector bundles is well-defined.

There is generalization of 4.1:

4.2 Definition. *Let $E \to X$ and $F \to Y$ be two vector bundles. A pair of continous maps* DLV

$$F : E \longrightarrow F, \quad X \longrightarrow Y$$

is called a bundle map, if the diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes and if the induced maps

$$E_a \longrightarrow F_{f(a)} \quad (a \in X)$$

are linear.

In the special case $X = Y$ and $f = \mathrm{id}$ one obtains 4.1 as special case. But the converse also is true if one uses the following construction: Let $f : X \to Y$ be a continuos map and $\pi_F : F \to Y$ a vector bundle on Y. One defines

$$E := X \times_Y F := \big\{\, (x, y); \quad f(x) = \pi_F(y) \,\big\}.$$

There is a natural projection $\pi_E : E \to X$ and this clearly is a vector bundle. There is also a natural map (projection) $F : E \to F$ and (f, F) is a bundle map in the sense of 4.1. In this special case the maps

$$E_a \longrightarrow F_{f(a)}$$

are isomorphisms of vector spaces. One calls $E \to X$ the pulled-back bundle of $F \to Y$ and uses the notation

$$f^*F := E \times YF.$$

It is easy to see that in general a bundle map

$$(f, F) : (E, X) \longrightarrow (F, Y)$$

in sense of 4.2 can be interpreted as bundle-map

$$E \to f^*F$$

in the sense of 4.1.

the definitions 4.1 and 4.1 have obvious variants in the differentiable real analytic and complex analytic worlds. We give an example of a differentiable bundle map:

Let $f : X \to Y$ be differentiable map of differentiable manifolds. For every $a \in X$ there is a natural homomorphism

$$\mathcal{C}^\infty_{Y,f(a)} \longrightarrow \mathcal{C}^\infty_{X,a}$$

which on the level of representatives is given by composition $g \longmapsto g \circ f$. If one composes a derivation $A \in T_aX$ with this homomorphism we get a derivation in $T_{f(a)}$. So we constructed a linear map

$$T_af : T_aX \longrightarrow T_{f(a)}Y.$$

This is called the **tangent map**. All tangent maps together yield a differentiable bundle map

$$(Tf, f) : (TX, X) \longrightarrow (TY, Y).$$

For (real or complex) analytic manifolds this map is (real or complex) analytic. by trivial reasons these constructions are compatible with the composition $X \to Y \to Z$.

It is worth while to describe the tangent map in local coordinates: Again let $f : X \to Y$ be differentiable map of differentiable manifolds and $a \in X$ a point. We choose charts φ around a in X and ψ around $f(a)$ in Y. We may assume $f(U_\varphi) \subset U_\psi$. Then f defines a differentiable map of open sets of coordinate spaces

$$f_{\varphi,\psi} : V_\varphi \longrightarrow V_\psi \qquad (V_\varphi \subset \mathbb{R}^n, \quad V_\psi \subset \mathbb{R}^m).$$

We can consider the Jacobimatrix $J(f_{\varphi,\psi})$ of this map. On the other side we recall that the space T_aX has a basis $[\partial/\partial x_1]_a, \dots, [\partial/\partial x_n]_a$ induced by the chart φ ans $T_{f(a)}$ has a basis $[\partial/\partial y_1]_{f(a)}, \dots, [\partial/\partial y_m]_{f(a)}$ induced by the chart ψ. If one uses these basises, the tangent map $T_aX \to T_{f(a)}$ is described by a matrix.

4.3 Proposition. *Let $f : X \to Y$ be differentiable map of differentiable manifolds, $a \in X$ a point, φ a chart around a in X and ψ a chart around $f(a)$ in Y. Then the matrix of the tangent map $T_aX \to T_{f(a)}Y$ with respect to the basises defined by φ and ψ is nothing else but the Jacobi matrix $J(f_{\varphi,\psi}, a)$.* TiJ

Proof. This is an application of the chain rule. (One has to know how the partial derivatives of a composition of maps are computed.) □

We mentioned that (by trivial reasons) the tangent map is compatible with composition of differentiable maps. This property is an abstract form of the chain rule. Hence 4.3 implies the usual chain rule for the Jacobi matrix. But this is not a honest new proof, because the chain rule also was buildt in the proof of 4.3. Also the theorems of invertible and implicit functions form calculus allow thanks to 4.3 abstract formulations:

4.4 Theorem of invertible functions. *Let $f : X \to Y$ be differentiable map of differentiable manifolds. Assume that the tangent map $T_aX \to T_{f(a)}Y$ is an isomorphism in some point a. Then the restriction of f to a suitable small open neighborhood of a is an open embedding.* TIf

This is a special case of

4.5 Theorem of invertible functions. *Let $f : X \to Y$ be differentiable map of differentiable manifolds. Assume that the tangent map $T_aX \to T_{f(a)}Y$ is injective for some point a. Then there exists a small open neighborhood U of a such that $f(U)$ is smooth an such that* TIF

$$U \xrightarrow{\sim} f(U)$$

is a diffeomorphism.

Both theorem hold in the (real and complex) analytic world.

5. Sections of vector bundles and differential forms

5.1 Definition. *Let $\pi : E \to X$ be a vector bundle. A section over an open subset $U \subset X$ is a continuous map $s : U \longrightarrow E$ with the property $\pi(s(x)) = x$ for all $x \in U$. In the case $U = X$ it is called a global section.* Dsv

A section cuts out on element of each fibre over. The set of all sections forms an module over the ring of all real-valued continuous functions. and over the ring of all complex-valued continuous functions if E is a complex vector bundle.

There are obvious variants in the differentiable and analytic worlds. We use the notation

$$E^\infty(U)$$

for the set of differentiable sections. This is a module over $\mathcal{C}^\infty(X, \mathbb{R})$ or $\mathcal{C}^\infty(X, \mathbb{C})$ depending on the fact whether we have a real or complex bundle.

An important example is the case of the tangent bundle TX of differentiable manifold. (Differentiable) sections of TX are called (differentiable) **vector fields**.

Important is the dual of the tangent bundle, the so-called cotangent bundle. There are several variants:

1. The *real cotangent bundle* T^*X of a differentiable manifold X is the bundle with the fibres
$$(TX)_a := \mathrm{Hom}_{\mathbb{R}}(T_aX, \mathbb{R}).$$
The differentiable sections of this bundle are called **real differentials**.

2. The *complex cotangent bundle* $T^*_{\mathbb{C}}X$ is the complex vector bundle with the fibres
$$(T^*_{\mathbb{C}}X)_a = \mathrm{Hom}_{\mathbb{R}}(T_aX, \mathbb{C}).$$
(These are complex vector spaces.) The differentiable sections of this bundle are called **complex differentials**.

3. Let X be an analytic manifold. We recall that the a priori real tangent space T_aX carries a structure as complex vector space and that the a priori real tangent bundle $TX \to X$ carries a structure as holomorphic bundle. The *holomorphic cotangent bundle* $T^*_{\text{hol}}X$ of X is the holomorphic bundle with the fibre
$$T^*_{\text{hol}}X = \text{Hom}_{\mathbb{C}}(T_aX, \mathbb{C}).$$
The holomorphic sections of this bundle are called **holomorphic differential forms**.

Hence holomorphic differentials are special complex differentials.

Vector fields as operators

Let A be a vector field on the differentiable manifold X. Let $f \in \mathcal{C}^\infty(U)$ be differentiable function on an open subset $U \subset X$. We can define a new function $Af : U \to \mathbb{C}$ by
$$(Af)(a) := A_a f_a.$$
The result of this construction is a new interpretation of vector fields. What one obtains is a so-called derivation on X.

5.2 Definition. *A derivation on a differentiable manifold X is an assignment to each open $U \subset X$ of a map* Ddx
$$A_U : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$
with the following properties:

1. *A_U is a derivation (i.e. $\mathbb{C}$-linear and $A_U(fg) = fA_U(g) + gA_U(f)$ is satisfied).*
2. *A_U is compatible with restriction to smaller open sets.*

The derivation is called real, if $\overline{A_U f} = A_U \bar{f}$.

the above construction assigns a derivation to any differentiable vector field A a derivation on X.

5.3 Remark. *The assignment of a derivation on X to a differentiable vector field on X is a one-to-one correspondence between differentiable vector fields and real derivations on X.* DeV

Dual fields as operators

let $E \to X$ be a differential vector bundle and Let A be a section over an open subset U and ω a section of the dual bundle E^* over U We define a function $\omega(A) : U \to \mathbb{R}$ by
$$\omega(A)(a) := \omega_a(A_a).$$
Thus differentials can operate on vector fields and the results are functions.

5.4 Remark. *Let $E \to X$ be a real differentiable vector bundle. Assume that there is an assignment of any open set U and any differentiable section A of E over U to a differentiable function* CDo

$$\omega_U(A) : U \to \mathbb{R}$$

such that the following conditions hold:

1. *This map is $\mathcal{C}^\infty(U)$-linear in A.*
2. *This map is compatible with restriction to smaller open subsets.*

Then there exists a unique differentiable section ω of E^* over X, such that

$$\omega_a(A_a) = \omega_U(A)(a).$$

The proof is trivial. □

Usually we will write $\omega(A)$ instead of $\omega_U(A)$. There are several generalizations of 5.4. Consider for example the bundle

$$\bigwedge^d E^*.$$

Recall that for a vector space V the exterior power $\bigwedge^d V$ was constructed as subset of $V^{\otimes}d$ and that this space can be identified with $\mathrm{Mult}(V^* \times \cdots \times V^*, \mathbb{R})$. Using this identification we obviously obtain

$$\bigwedge^d V = \mathrm{Alt}(V_1^* \times \cdots \times V_d^*, \mathbb{R}),$$

where $\mathrm{Alt}(\cdot)$ denotes the subspace of alternating multilinear forms of $\mathrm{Mult}(\cdot)$. Going to the dual and using $V = V^{**}$ we obtain

$$\bigwedge^d V^* = \mathrm{Alt}(V \times \cdots \times V, \mathbb{R}).$$

Now a straightfoward generaliization of 5.4 says:

5.5 Remark. *Let $E \to X$ be a real differentiable vector bundle. The differentiable sections of $\bigwedge^d E^*$ over X are in one-to-one correspondence to assignments (ω_U) of any open set U to a map, which assigns d differentiable sections $A_1, \ldots, A_d$ of E over U a differentiable function* CdA

$$\omega_U(A_1, \ldots, A_d) : U \longrightarrow \mathbb{R},$$

such that the following conditions hold:

1. *ω_U is $\mathcal{C}^\infty(U)$-multilinear in the A-s*
2. *It is also alternating in the A-s.*
3. *This assignment is compatible with restriction to smaller open subsets.*

One may ask whether bundle maps can be seen as an operation on sections. Let $(F, f) : (E, X) \to (F, Y)$ be a differentiable bundle map of differentiable vector bundles. Then for every point $a \in X$ there is an induced linear map $E_a \to F_{f(a)}$. We can consider the dual of this map ore more generally

$$\bigwedge^d F^*_{f(a)} \longrightarrow \bigwedge^d E^*_a.$$

5.6 Remark. *Let $(E, X) \to (F, Y)$ be a differentiable bundle map of real vector bundles. Let $V \subset Y$ be open. There is a natural "pull-back" map* DPs

$$\Big(\bigwedge^d E^*\Big)^\infty (V) \longrightarrow \Big(\bigwedge^d F^*\Big)^\infty (f^{-1}(V)).$$

There are obvious variants of this construction, which we will use without much comments: There is a variant for complex bundles and there are analytic variants. There is also the following variant. One starts with real bundles E, F but one replaces E^* (similarly F^*) by the complex bundle with the fibres $\mathrm{Hom}_{\mathbb{R}}(E_a, \mathbb{C})$.

A special case of this construction is the pull-back of differential forms: Let $f : X \to Y$ be a differentiable map of differentiable manifolds. We consider the induced tangent map

$$(TX, X) \longrightarrow (TY, Y).$$

From 5.6 we know that differential forms can be pulled back. This is possible for real- and complex valued differential forms.

6. The calculus of differential forms.

Let X be a differentiable manifold. In the following we denote by $A^d(U)$ the space of all differentiable alternating forms over an open subset $U \subset X$. At the moment it doesn't matter whether we consider real or complex differential forms. We agree that $A^d(U)$ means the one or the other. It is enough to treat the case $U = X$, because open subsets can be considered as differentiable manifolds as well. We collect the operations of the previous section and add one more:

1. $A^d(X)$ is a module over the ring of differentiable functions. In the case $d = 0$ it equals the ring of differentiable functions. On has $A^d(X) = 0$ for $d < 0$ and $d > \dim X$.
2. There is a "skew product"

$$A^p(X) \times A^q(X) \longrightarrow A^{p+q}(X)$$

On the case $p = 0$ the skew multiplication is simply the standard multiplication with functions.

3. The skew product is associative and skew commutative. The latter means

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \qquad (\alpha \in A^p(X),\ \beta \in A^q(X)),$$

especially

$$\omega \wedge \omega = 0 \quad \text{for odd } d \qquad (\omega \in A^d(X)).$$

From the associativity follows that $\omega_1 \wedge \ldots \wedge \omega_m$ is defined.

4. We introduce a new operation. the exterior differentiation. Here we make use of the fact that differential forms can be considered as alternating multilinear forms on vector fields (5.6) and that vector fields operate on functions (5.3).

6.1 Definition. *The exterior differentiation* DaA

$$d : A^d(X) \longrightarrow A^{d+1}(X)$$

is defined by

$$\begin{aligned}(d\omega)(A_1, \ldots, A_{d+1}) := &\sum_{i=1}^{d+1} (-1)^{i+1} A_i \omega(A_1, \ldots, \hat{A}_i, \ldots, A_{d+1}) + \\ &\sum_{i<j} (-1)^{i+j} \omega([A_i, A_j], A_1, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_{d+1}).\end{aligned}$$

One can check that this is alternating and multilinear over the ring of differentiable functions and hence defines a differential form. This formula will be clearer in the local version.

5. The exterior differentiation is a vector space homomorphism which satisfies

$$d \circ d = 0.$$

6. The following product rules hold: For functions one has

$$d(fg) = fd(g) + gd(f)$$

or more general for differential forms

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d(\beta) + d(\alpha) \wedge \beta \qquad (\alpha \in A^p(X)).$$

As a consequence one has

$$d(\omega_1) = 0, \ldots, d(\omega_m) = 0 \Longrightarrow d(\omega_1 \wedge \ldots \wedge \omega_m) = 0.$$

A special case is also

$$d(df_1 \wedge \ldots \wedge df_m) = 0.$$

7. There is a pull-back map for a differentiable map $f : X \to Y$:

$$f^* : A^d(Y) \longrightarrow A^d(X).$$

In the case $d = 0$ this is the usual composition of maps. The pull-back is a vector space homomorphism and even more there are the following compatibilities:

$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta), \quad f^*(d\omega) = df^*(\omega).$$

8. All these constructions are compatible with restrictions to open submanifolds.

A differential form ω on X is known, if its restrictions to the members U_i of an open covering is known. Hence the whole calculus is regulated locally. Using charts this means that it is enough to know the calculus for open subsets $U \subset \mathbb{R}^n$. We reformulate the calculus in this case:

Recall that $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are basis vector fields on U. Every vector field can be written as linear combinations of them using differentiable functions as coefficients:

6.2 Definition. *For an open subset $U \subset \mathbb{R}^n$ we define the differentials* DbD

$$dx_1, \ldots, dx_n$$

by

$$dx_i(\partial/\partial x_j) = \delta_{ij}.$$

Then every differential can be written in the form

$$\omega = f_1 dx_1 + \cdots + f_n dx_n.$$

More general every element $\omega \in A^d(U)$ has a unique representation of the form

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_d} f_{i_1,\ldots,i_d} dx_{i_1} \wedge \ldots \wedge dx_{i_d}$$

with differentiable functions $f_{\ldots}$.

The alternating product is regulated by the conditions that it is distributive and associative and that

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \qquad (\Longrightarrow dx_i \wedge dx_i = 0).$$

The exterior differentiation of a function is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

and more general for forms by

$$d \sum_{1 \le i_1 < i_2 < \cdots < i_d} f_{i_1,\ldots,i_d} dx_{i_1} \wedge \ldots \wedge dx_{i_d} = \sum_{1 \le i_1 < i_2 < \cdots < i_d} df_{i_1,\ldots,i_d} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_d}.$$

Let $V \mathrm{subset} \mathbb{R}^{\mathrm{m}}$ be another open subset and $U \to V$ a differentiable map. The pullback is regulated by

$$f^*(dy_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dxj \qquad (1 \le i \le m).$$

Notice that this follows from the compatibility $f^*(dg) = d(f^*(g))$ applied to the projection $g(y) = y_i$.

Differential forms on complex analytic manifolds

In the case of a complex analytic manifold X the above constructions leads to a refinement. Firstly we notice that a X is also a differentiable manifold, hence $A^d(X)$ is defined. In this context we consider the complex variant of $A^d(X)$. Hence it is a module over the ring of complex valued differentiable functions $\mathcal{C}^\infty(X)$. Recall that this means that we consider sections of the bundle

$$\bigwedge^d \mathrm{Hom}_{\mathbb{R}}(T_a X, \mathbb{C}).$$

But now, since X is a complex analytic manifold, the space $T_a X$ has a structure as complex vector space. This leads to the announced refinements. To make this clear, we do a little linear algebra.

Let V be a *complex* vector space. Then on the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ we have a natural conjugation $L \mapsto \bar{L}$. It is defined by

$$\bar{L}(v) : \overline{L(v)}.$$

(Notice that on an abstract complex vector space complex conjugation is not well-defined.) We consider the subspace od $\mathbb{C}$-linear maps

$$\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}).$$

A dimension consideration shows

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})}.$$

Elements of $\overline{\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})}$ are so-called $\mathbb{C}$-antilinear maps. they follow the rule $L(Cv)=\bar{C}L(v)$. So we have to understand the Grassmann algebra of a direct sum $A\oplus B$ of two (in our case complex) vector spaces. There is a linear map

$$\bigwedge^p A\otimes\bigwedge^q B\longrightarrow\bigwedge^d(A\oplus B)\qquad(d=p+q),$$

which sends $\alpha\otimes\beta$ to $\alpha\wedge\beta$. The image of this map is denoted by

$$\bigwedge^{p,q}(A\oplus B)\subset\bigwedge^d(A\oplus B).$$

For example by means of basises it is easy to check: The map

$$\bigwedge^p A\otimes\bigwedge^q B\stackrel{\sim}{\longrightarrow}\bigwedge^{p,q}(A\oplus B)$$

is an isomorphism. On has the direct sum decomposition

$$\bigwedge^d(A\oplus B)=\bigoplus_{p+q=d}\bigwedge^{p,q}(A\oplus B).$$

In our case

$$\bigwedge^d\mathrm{Hom}_{\mathbb{R}}(T_aX,\mathbb{C}))=\bigoplus_{p+q=d}\bigwedge^p\mathrm{Hom}_{\mathbb{C}}(T_aX,\mathbb{C})\otimes\bigwedge^q\overline{\mathrm{Hom}_{\mathbb{C}}(T_aX,\mathbb{C})}.$$

A differential form ω is called of type (p,q) if

$$\omega_a\in\bigwedge^{p,q}\mathrm{Hom}_{\mathbb{R}}(T_aX,\mathbb{C}))$$

for all points a. We denote by

$$A^{p,q}(X)\subset A^{p+q}(X)$$

the subspace of all forms of type (p,q). There is a decomposition

$$A^d(X)=\bigoplus_{p+q=d}A^{p,q}(X).$$

The wedge product preserves this graduation, i.e. it defines maps

$$A^{p,q}(X)\times A^{p',q'}(X)\stackrel{\wedge}{\longrightarrow}A^{p+p',q+q'}(X).$$

But the total derivative d does not preserve it. To remedy this situation we observe that there are projections

$$A^{p+q}(X)\longrightarrow A^{p,q}(X).$$

6.3 Definition and remark. *Let X be a complex analytic manifold. The composition of d with the natural projections gives operators* DDb

$$\partial:A^{p,q}(X)\longrightarrow A^{p+1,q}(X),\quad\bar\partial:A^{p,q}(X)\longrightarrow A^{p,q+1}(X).$$

These operators satisfy

$$d=\partial+\bar\partial\qquad(on\ A^{p,q}(X))$$

and

$$\partial\circ\partial=0,\quad\bar\partial\circ\bar\partial=0,\quad\partial\circ\bar\partial=-\bar\partial\circ\partial.$$

Finally we mention $\overline{\partial\omega}=\bar\partial\bar\omega$.

We express the complex calculus in local coordinates: Let $U \subset \mathbb{C}^n$ be an open subset. We define

$$dz_i = dx_1 + \mathrm{i}dy_i, \quad d\bar{z}_i = dx_i - \mathrm{i}dy_i.$$

We claim that dz_i is of type $(1,0)$ (and similarly that $d\bar{z}_i$ is of type $(0,1)$. First we notice

$$dz_i\Big(\frac{\partial}{\partial x_j}\Big) = \delta_{ij} \quad \text{and} \quad dz_i\Big(\frac{\partial}{\partial y_j}\Big) = \mathrm{i}\delta_{ij}.$$

Using 2.6 we see

$$dz_i\Big(J\frac{\partial}{\partial x_j}\Big) = \mathrm{i}dz_i\Big(\frac{\partial}{\partial x_j}\Big)$$

and the same for $\partial/\partial y_i$ instead of $\partial/\partial x_i$. This shows the $\mathbb{C}$-linearity of dz_i, which means that it is of type $(1,0)$.

6.4 Proposition. *Let $U \subset \mathbb{C}^n$ be an open subset. Elements of $A^{p,q}(U)$ have a unique representation in the form* KaC

$$\omega = \sum_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} f_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} dz_{i_1}\wedge\ldots\wedge dz_{ip}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge d\bar{z}_{j_q}.$$

The operators ∂ and $\bar{\partial}$ are given for functions by

$$\partial f = \sum_{i=1}^n \frac{\partial}{\partial z_i}\,dz_i, \quad \bar{\partial} f = \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i}\,d\bar{z}_i$$

and on forms by

$$\bar{\partial}\sum_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} f_{\substack{i_1,\ldots,i_p\\ j_1,\ldots,j_q}} dz_{i_1}\wedge\ldots\wedge dz_{ip}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge d\bar{z}_{j_q} =$$
$$\sum_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} \bar{\partial} f_{\substack{i_1,\ldots,i_p\\ j_1,\ldots,j_q}} \wedge dz_{i_1}\wedge\ldots\wedge dz_{ip}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge d\bar{z}_{j_q}.$$

and similarly for ∂ instead of $\bar{\partial}$.

We conclude this section with some remark about holomorphic differential forms. We know already what they are, namely holomorphic sections of the holomorphic cotangent bundle. The fibres of this bundle are $\mathrm{Hom}_{\mathbb{C}}(T_a, \mathbb{C})$. Hence holomorphic differential forms are special kinds of forms of type $(p,0)$. It is easy to verify

6.5 Lemma. *A differential form ω on a complex analytic manifold is holomorphic if and only if it satisfies the following two conditions:* LcH

a) *It is of type* $(p, 0)$
b) $\bar{\partial}\omega = 0$.

Especially $d\omega = \partial\omega$ for holomorphic forms.

We denote by

$$\Omega^m(X)$$

the space of all holomorphic differential forms of degree m on X.

Chapter III. Hodge theory

1. The lemmas of Poincarè and de Rham

Let X be a differentiable manifold: The sequence

$$\cdots \longrightarrow A^{m-1}(X) \xrightarrow{d} A^m(X) \longrightarrow A^{m+1} \xrightarrow{d} \cdots$$

is called the de-Rham complex. Later we will introduce complexes an cohomology in a general context, but this is not necessary at the moment. For convenience we use the notion of an exact sequence already here: A sequence (finite or not) of abelian groups

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

is called exact of for every i

$$\text{Kernel}(A_i \longrightarrow A_{i+1}) = \text{Image}(A_{i-1} \longrightarrow A_i).$$

We know that $d \circ d = 0$. A differential form ω is called **closed** if $d\omega = 0$ and **total** if it is of the form $\omega = d\omega'$. Hence total differential forms are closed. We are interested in converse results. i.e. in conditions under which the de-Rham is exact. For this purpose we introduce the **de-Rham cohomology**, which measures the non-exactness. We set

$$H^m_{\text{dR}}(X) := \frac{\text{Kernel}(A^m(X) \longrightarrow A^{m+1}(X))}{\text{Image}(A^{m-1}(X) \longrightarrow A^m(X))}.$$

Recall that we can consider differential forms real- and complex-valued. To indicate the difference we use the notations

$$H^m_{\text{dR}}(X, \mathbb{R}), \quad H^m_{\text{dR}}(X, \mathbb{C}).$$

There is a natural map

$$H^m_{\text{dR}}(X, \mathbb{R}) \longrightarrow H^m_{\text{dR}}(X, \mathbb{C})$$

and one easily shows that this map is injective and that

$$H^m_{\mathrm{dR}}(X,\mathbb{C}) = H^m_{\mathrm{dR}}(X,\mathbb{R}) \oplus \mathrm{i}\, H^m_{\mathrm{dR}}(X,\mathbb{R}).$$

Hence the difference is not big. Of course

$$H^m_{\mathrm{dR}}(X,\mathbb{C}) = 0 \quad \text{for} \quad m < 0 \text{ and } m > \dim X.$$

The space $H\mathrm{dR}^0(X,\mathbb{C})$ can be identified with all functions from $\mathcal{C}^\infty(X)$, which are annihilated by d. These are the locally constant functions. When X is connected we see

$$H^0_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathbb{C} \qquad (\text{and} \quad H^0_{\mathrm{dR}}(X,\mathbb{R}) \cong \mathbb{R}).$$

The higher cohomology groups are involved. A basis result is:

1.1 Lemma of Poincarè. *Let U be an open convex subset of $\mathbb{R}^n$. The sequence* LoP

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(U) \longrightarrow \cdots \longrightarrow A^n(U) \longrightarrow 0$$

is exact. Here $\mathbb{R} \longrightarrow A^0(U)$ means the map which assigns to a real number the corresponding constant function. Hence $H^0_{\mathrm{dR}}(U,\mathbb{R}) = \mathbb{R}$ and

$$H^m_{\mathrm{dR}}(U,\mathbb{R}) = 0 \quad \textit{for} \quad m > 0,$$

i.e. every closed differential form is total.

The lemma of Poincarè has a holomorphic version, which we don't need but which we formulate just for the sake of completeness

1.2 Holomorphic lemma of Poincarè. *Let $U \subset \mathbb{C}^n$ be an open convex subset. the sequence* HLp

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \Omega^1(U) \longrightarrow \cdots \longrightarrow \Omega^n(U) \longrightarrow 0$$

is exact.

The lemma of Poincarè has another complex version, which is fundamental for us: The so-called Dolbeault-complex or $\bar\partial$-complex on a complex analytic manifold is

$$\cdots \longrightarrow A^{p,q-1}(X) \xrightarrow{\bar\partial} A^{p,q}(X) \longrightarrow A^{p,q+1} \xrightarrow{\bar\partial} \cdots$$

Because of $\bar\partial \circ \bar\partial = 0$ we can define the **Dolbeault-cohomology**

$$H^{p,q}(X) := \frac{\mathrm{Kernel}(A^{p,q}(X) \longrightarrow A^{p,q+1}(X))}{\mathrm{Image}(A^{p,q-1}(X) \longrightarrow A^{p,q}(X))}.$$

1.3 Lemma of Dolbeault. *Let U be an open convex subset of $\mathbb{C}^n$. The sequence* LVd

$$0 \longrightarrow \Omega^p(U) \longrightarrow A^{p,q}(U) \longrightarrow A^{p.q+1}(U) \longrightarrow \cdots \longrightarrow A^{p,n-p}(U) \longrightarrow 0$$

is exact. Hence $H^{p,0}(U) = \Omega(U)$ and

$$H^{p,q}(U) = 0 \quad \textit{for} \quad q > 0.$$

1.4 Proposition. *Sei $U \subset \mathbb{C}^n$ an open convex subset and* DDf

$$\alpha \in A^{1,1}(U) \cap A^2_{\mathbb{R}}(U), \quad d\alpha = 0.$$

Then

$$\alpha = \mathrm{i}\partial\bar{\partial} f$$

with a real differentiable function f

Proof. From the lemma of Poincarè we know $\alpha = d\beta$. We decompose $\beta = \gamma + \bar{\gamma}$ with $\gamma \in A^{1,0}$. We have $\partial\gamma = 0$ and, using the lemma of Dolbeault, $\gamma = \partial h$. Set $f = \mathrm{i}(h - \bar{h})$. □

2. Elliptic differential operators

Let $U \subset \mathbb{R}^n$ be an open subset. We are interested in maps

$$D : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

which can be written as finite sum

$$Df = \sum h_{i_1,\dots,i_m} \frac{\partial^{i_1+\cdots+i_m} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$$

with differentiable coefficients $h_{\dots}$, which are uniquely determined. We call D a local linear differential operator. When D is non-zero there exists a maximal m such that $h_{i_1,\dots,i_n}$ is non-zero for some index with $i_1 + \cdots + i_n = m$. We call m the degree of this operator and the function on $U \times \mathbb{R}^n$

$$P(x_1, \dots, x_n, X_1, \dots, X_n) = \sum_{i_1+\cdots+i_n=m} h_{i_1,\dots,i_m}(x) X_1^{i_1} \dots X_n^{i_n}$$

is called the **symbol** of D. This is homogenous polynomial of degree m for fixed x. The operator D is called elliptic, if

$$P(x, X) \neq 0 \quad \text{for all} \quad X \neq (0, \dots, 0).$$

We need a slight generalization of this. We consider operators

$$D : \mathcal{C}^\infty(U)^p \longrightarrow \mathcal{C}^\infty(U)^q.$$

They can be considered as $p \times q$ matrices as well as the coefficients The degree now is defined to be the biggest number m such that one of the coefficients of $h_{i_1,\dots,i_n}$ is different from zero for some index with $i_1 + \cdots + i_n = m$. The symbol now is a $p \times q$ matrix of functions on $U \times \mathbb{R}^n$.

2.1 Definition. *A local linear differential operator* DeO

$$D : \mathcal{C}^\infty(U)^p \longrightarrow \mathcal{C}^\infty(U)^q \qquad (U \subset \mathbb{R} \text{ open})$$

is called ***elliptic****, if* $p = q$ *and if the symbol* $P(x, X)$ *is an invertible matrix for every* $x \in D$ *and every* $X \neq 0$.

Example. The Laplace operator

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We want to generalize the notion of an elliptic operator on manifolds with vector bundles. Let E, F be two differentiable vector bundles over a differentiable manifold. Recall that we denote by $E^\infty(U)$ the space of differentiable sections of E over an open subset $U \subset X$. By E_U we denote the inverse image of U in E. This is bundle over U,

$$E_U \longrightarrow U.$$

Similar notations are used for F.

2.2 Definition. *An linear differential operator* $D : E \to F$ *by definition is a family of operators* DeB

$$D : E^\infty(U) \longrightarrow F^\infty(U) \qquad (U \subset X \text{ open})$$

with the following properties:

1. *D is compatible with restriction to smaller open sets.*
2. *For every point $a \in X$ there exist an open neighborhood $a \in U \subset X$ and local trivializations*

$$E_U \xrightarrow{\sim} U \times \mathbb{R}^p, \quad F_U \xrightarrow{\sim} U \times \mathbb{R}^q$$

such that the induced maps $\mathcal{C}^\infty(U)^p \to \mathcal{C}^\infty(U)^q$ *are local linear differential operators. D is called elliptic, if an can achieve that they are are elliptic in the sense of 2.1.*

One of the basic facts about elliptic operators D on a *compact* differentiable manifold the kernel and the kokernel of $D : E^\infty(X) \to F^\infty(X)$ is finite dimensional. (The kokernel of a linear map $L : V \to W$ is defined as $W/L(V)$.) This and a refinement will be formulated as a theorem a little later:

2.3 Definition. *Let E be a real vector bundle on a differentiable manifold X. A Euclidean metric on E is a family of symmetric positive definit bilinear forms* DRm

$$E_a \times E_a \longrightarrow \mathbb{R},$$

which depends differentiable on a.

2.4 Definition. *Let E be a complex vector bundle on a differentiable manifold X. A Hermitean metric on E is a family of symmetric positive definit Hermitian forms* DRm

$$E_a \times E_a \longrightarrow \mathbb{C},$$

which depends differentiable on a.

In both cases a pairing

$$E^\infty(U) \times E^\infty(U) \longrightarrow \mathcal{C}^\infty(U), \quad (s,t) \longmapsto \langle s,t\rangle$$

is induced.

Two linear differential operators

$$D : E \longrightarrow F, \quad D^* : F \longrightarrow E$$

are called **formally adjoint**, if for every open subset $U \subset X$ and sections $s \in E^\infty(U)$, $t \in F^\infty(U)$ the formula

$$\langle Ds, t\rangle = \langle s, D^* t\rangle$$

holds. It is not difficult to check:

2.5 Remark. *Let X be a differentiable manifold and let $E \to X$, $F \to X$ be real or complex differentiable vector bundles with Euclidean or Hermitian metric. Every linear differential operator $D : E \to F$ admits a unique formally adjoint $D^* : F \to E$ and this is elliptic when D is elliptic. The formally adjoint of D^* is D.* DaO

We formulate now without proof a fundamental result of the theory of partial differential equations:

2.6 Theorem. *Let E, F be real or complex vector bundles with Euclidian or Hermitian metric a compact differentiable manifold. Let $D : E \to F$ be an elliptic operator. Then the kernel and kokernel of the map* HPD

$$D : E^\infty(X) \longrightarrow F^\infty(X)$$

are finite dimensional. Moreover

$$E^\infty(X) = \mathrm{Kernel}\big(D : E^\infty(X) \to F^\infty(X)\big) \oplus \mathrm{Image}\big(D^* : F^\infty(X) \to E^\infty(X)\big).$$

3. Integration

We need the notion of **orientation** of a finite dimensional real vector space V. Two basises of V are called orientation compatible if the base transition matrix has positive determinant. There are two The set of all basises decomposes into two classes in which each two are orientation compatible. An orientation of V is the choice of one of the two. They are then called oriented baises. Hence every basis induces an orientation and two basises define the same orientation if the are orientation compatible. The standard orientation of $\mathbb{R}^n$ is defined by the standard basis. Let $L : V \to W$ be an isomorphism of oriented vector spaces. This ismorphism os called orientation preserving if oriented basis are mapped to oriented basises.

3.1 Definition. *An orientation of a differentiable manifold X is a choice of an orientation on each tangent space T_aX, such that the following condition is satisfied: The manifold X can be covered by charts $\varphi : U_\varphi \longrightarrow V_\varphi$ with the property that the tangent maps* DoM

$$T_aU_\varphi \longrightarrow T_{\varphi(a)}V_\varphi = \mathbb{R}^n$$

are orientation preserving. The charts with this property are called ***oriented charts***

Let φ, ψ be two oriented charts. Then the functional matrix of $\psi\varphi^{-1}$ has everywhere positive determinant. Let conversely an atlas be given, such that every two charts from this atlas have this property, then there exists an orientation of X, such that the charts of this atlas are orientable.

Integration

Let X be an oriented differentiablemanifold of pure dimension n. Let ω be a differential form. The support is defined

$$\operatorname{supp}(f) := \overline{\{\, a \in X; \quad \omega_a \neq 0 \,\}}.$$

We denote by $\mathcal{A}_c^n(X)$ the set of compactly supported top-differential forms. The support of an $\omega \in \mathcal{A}_c^n(X)$ is called small, if there exists an oriented chart φ with $\operatorname{supp}(\omega) \subset U_\varphi$. The form ω can be written in this chart as $f(x)dx_1 \wedge \ldots \wedge dx_n$. We define

$$\int_X \omega := \int_{V_\varphi} f(x)\, dx_1 \ldots dx_n.$$

From the transformation formula for integrals one can see that this definition is independent from the choice of φ. (Here one has to use that the chart transformations have positive Jacobi determinant.)

Using the technique of decomposition of 1, one can show:
There exists a unique linear form

$$A^n_c(X) \longrightarrow \mathbb{C}, \quad \omega \longmapsto \int_X \omega,$$

which agrees with the above construction for forms with small support.
This integral can be extended by the standard techniques of integration theory to a large class of even not continuous differential forms and one uses this technique also to define $\int_A \omega$ for subsets $A \subset X$. We need only little of these constructions, for example we will use that $\int_U \omega$ can be defined for open subsets and differential forms $\omega \in A^n(X)$ such that $\operatorname{supp}(\omega) \cap \bar{U}$ is compact.

The theorem of Stokes

Let $U \subset X$ an open subset of an oriented differentiable manifold. Let a be a boundary point of U. We say that a is a smooth boundary point of U, if there is an oriented chart φ around a such that

$$\varphi(U \cap U_\varphi) = \{x \in V_\varphi,\ x_n > 0\}.$$

Then automatically

$$\varphi(U \cap \partial U_\varphi) = \{x \in V_\varphi,\ x_n = 0\}.$$

We denote by $\partial_0 U$ the smooth part of the boundary. It is clear that $\partial_0 X$ is a smooth subset and hence a differentiable manifold. A chart around a is given by the restriction of φ when we consider $\{x \in V_\varphi,\ x_n = 0\}$ as an open subset of $\mathbb{R}^{n-1}$. It can be checked that there is an orientation on ∂_U such that these charts are orientable. The theorem of Stokes states:
Let $\omega \in A^{n-1}(X)$ be a differential form of degree $n-1$ such that

$$\operatorname{supp}(\omega) \cap \bar{U}$$

is compact. Then

$$\int_U d\omega = \int_{\partial U} \omega \mid \partial U.$$

A special case says that for compact X and arbitrary $\omega \in A^{n-1}(X)$

$$\int_X d\omega = 0$$

(because one take $U = X$ with $\partial U = \emptyset$.)

4. Real Hodge theory

4.1 Definition. *A Riemannian manifold is a differential manifold together with a Euclidean metric g on the tangent bundle.* DRm

If $U \subset \mathbb{R}^n$ is an open subset, than a Riemannian metric on U is given by a $n \times n$-matrix $g(x) = (g_{ik}(x))$ of differentiable fucntions, which is symmetric and positiv definit at every point. (Identfy the tangent space with $\mathbb{R}^n$.)

We need some more linear algebra. Let V be an oriented real vector space of dimension n. An element of the one-dimensional space $\bigwedge^n V$ is called positive if it is of the form $Ce_1 \ldots \wedge \ldots e_n$, $C > 0$, with an oriented basis. It is clear that this definition is independent of the choice of the oriented a basis. Another way to express this, is to say that $\bigwedge^n V$ has been oriented. Now let a also Euclidian metric $\langle \cdot, \cdot \rangle$ one V be given. It is possible to extend this to Euckidean metrics on all $\bigwedge^d V$ by the such that the following condition is satisfied:

$$\langle a_1 \wedge \ldots \wedge a_d, b_1 \wedge \ldots \wedge b_d \rangle = \det\bigl(\langle a_i, b_j \rangle\bigr)_{1 \le i,j \le n}.$$

In the top space $\bigwedge^n V$ we now can chose the unique *positive* Element A with the property $\langle A, A \rangle = 1$. We call this the *fundamental element.* The fundamental element defines an isomorphism

$$\mathbb{R} \xrightarrow{\sim} \bigwedge^n V, \quad C \longmapsto CA.$$

We apply this to an oriented Riemannian manifold. (X, g). The tangent space T_aX then carries a Euclidean metric. This defines also an isomorphism $T_aX \to T_aX^*$. We use this isomorphism to carry over the Euclidean metric to a Euclidean metric on T_aX^*. The dal space of an oriented space is oriented as well (by the dual basises of the oriented basises. Hence we can consider the fundamental element of the top space $\bigwedge^n(T_aX^*)$. This defines a top-differential form on X.

4.2 Remark. *Let (X, g) be an oriented Riemannian manifold of pure dimension n. The fundamental element in $\bigwedge^n(T_aX^*)$ defines a top-differential form $\omega \in A^n(X)$. Im the case of an open subset $U \subset \mathbb{R}^n$ it is given by* EFf

$$\sqrt{\det g(x)}\, dx_1 \wedge \ldots, \wedge dx_n.$$

This form is called the ***volume form*** *of X. Its integral (it can be infinite) is called the volume of X.*

The star operator

If V is an oriented real vector space of dimension n then we defined an isomorphism $\bigwedge^n V \cong \mathbb{R}$. Hence we obtain a pairing

$$\bigwedge^p V \times \bigwedge^{n-q} V \longrightarrow \mathbb{R}, \quad (A, B) \longmapsto A \wedge B.$$

This pairing is non-degenerated and induces an isomorphism

$$\bigwedge^p V \xrightarrow{\sim} \Bigl(\bigwedge^{n-q} V\Bigr)^*.$$

As we mentioned already the Euclidean metric extends to the exterior powers. Hence we obtain an Isomorphismus, the so-called **star operator**

$$\bigwedge^p V \xrightarrow{\sim} \bigwedge^{n-q} V, \quad A \longmapsto *A.$$

This construction extends to oriented Riemannian manifolds:

4.3 Remark. *Let (X, g) be an oriented Riemannian manifold of pure dimension n. The star operators for tangent spaces induce an isomorphism* DSo

$$A^p(X) \xrightarrow{\sim} A^{n-p}(X).$$

This has the properties

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega,$$

where ω denotes the volume form and

$$* * \alpha = (-1)^{p(n-p)} \alpha \quad \textit{for} \quad \alpha \in A^p(X).$$

Here $\langle \cdot, \cdot \rangle$ comes from the scalar product on $\bigwedge^p (T_a X)^*$ and hence $\langle \alpha, \beta \rangle$ is a function. But we can integrate this function to get a pairing, which produces scalars, more precisely:

Let (X, g) be a compact oriented Riemannian manifold. Then one can define

$$A^p(X) \times A^p(X) \longrightarrow \mathbb{R}, \quad (\alpha, \beta)_g := \int_X \langle \alpha, \beta \rangle\, \omega = \int_X \alpha \wedge *\beta.$$

This is a symmetric positive definit bilinear form.

The codifferentiation

We define the codifferentiation by

$$d^* : A^p(X) \longrightarrow A^{p-1}(X), \quad d^* = -(-1)^{n(p+1)} * d * .$$

4.4 Proposition. *The codifferentiation on a compact oriented Riemannian manifold satisfies* CiA

$$(d\alpha, \beta) = (\alpha, d^*\beta), \quad \alpha \in A^{p-1}(X),\ \beta \in A^p(X),$$

hence d^* is the adjoint for d.

Proof. One has to use Stokes formula $\int_X d(\alpha \wedge *\beta) = 0$ and the product rule. □

The Laplace-Beltrami operator

The Laplace-Beltrami operator on an oriented Riemannian manifold X is defined by

$$\Delta : A^p(X) \longrightarrow A^p(X), \quad \Delta = dd^* + d^*d.$$

The simplest case is the Euclidean metric on an open subset $U \subset \mathbb{R}^n$ and the case $p = 0$. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_1^2}.$$

This example makes plausible:

4.5 Remark. *The Laplace-Beltrami operator can be considered as linear differential operator of The bundle $\bigwedge^p TX^*$ into itself. This operator is elliptic and it is its own adjoint.* LPe

We denote by

$$\mathcal{H}^p(X) = \left\{ \alpha \in A^p(X); \quad \Delta\omega = 0 \right\}$$

the kernel of Δ. Its elements are called **harmonic forms**. Now we can apply the theory of partial differential equations to conclude in case of a compact oriented Riemannian manifold X:

$$A^p(X) = \mathcal{H}^p(X) \oplus \Delta(A^p(X)).$$

In the case of compact manifolds are easy to characterize:

4.6 Proposition. *A differential form α on a compact oriented Riemannian manifold X is harmonic if and only if* RHd

$$d\alpha = 0 \qquad \text{and} \qquad d^*\alpha = 0.$$

If X is connected then every harmonic function (=zero-form) is constant.

The proof follows from

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha). \qquad \square.$$

As a consequence of 4.6 we obtain for $\alpha \in A^p(X)$ a representation

$$\alpha = \alpha_0 + d\beta + d^*\gamma, \qquad \alpha_0 \text{ harmonic.}$$

We apply this to closed forms α. From $d\alpha = 0$ and 4.6 follows $dd^*\gamma = 0$, hence

$$(d^*\gamma, d^*\gamma) = (\gamma, dd^*\gamma).$$

It follows

$$\alpha = \alpha + d\beta$$

and this is a direct decomposition

$$\text{Kernel}(A^p(X) \longrightarrow A^{p+1}(X)) = \mathcal{H}^p \,\oplus\, \text{Image}(A^{p-1}(X) \longrightarrow A^p(X)).$$

This means that every class of closed forms in $H^p_{\text{dR}}(X, \mathbb{R})$ contains a unique harmonic representant. This means:

4.7 Main theorem of real Hodge theory. *Let X be a compact oriented Riemannian manifold. Then $\mathcal{H}^p(X)$ is contained in the space of closed forms and the natural homorphism* MTH

$$\mathcal{H}^p(X) \xrightarrow{\sim} H^p_{\text{dR}}(X, \mathbb{R})$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called **Betti-numbers**

$$b^p(X) := \dim H^p_{\text{dR}}(X, \mathbb{R})$$

ere well defined numbers.

As an application we derive the duality theorem. When α is harmonic then by trivial reasons $*\alpha$ is harmonic too. This obviously defines an isomorphism

$$\mathcal{H}^p(X) \xrightarrow{\sim} \mathcal{H}^{n-p}(X).$$

4.8 Poincarè duality. *Let X be a pure n-dimensional compact oriented Riemannian manifold. The pairing $(\cdot, \cdot)$* PD

$$H^p_{\text{dR}}(X, \mathbb{R}) \times H^{n-p}_{\text{dR}}(X, \mathbb{R}) \longrightarrow \mathbb{R}, \qquad \int_X \alpha \wedge \beta,$$

is non-degenerated, hence

$$b^p(X) = b^{n-p}(X).$$

5. Complex Hodge theory

Again we start with a little linear algebra. A Hermitean form h on a complex vector space V is a map $V \times V \to \mathbb{C}$ which is linear in the first variable and such that

$$h(a,b) = \overline{h(b,a)}.$$

The form h is called positive definit if (the real expression) $h(a,a)$ is positive for non-zero h. A positive Hermitian form is called a Hermitean metric. It is clear that

$$g(a,b) := \operatorname{Re} h(a,b) = \frac{1}{2}(h(a,b) + h(b,a))$$

is a bilinear form on the underlying real vector space. Hence a Hermitian metric has an underlying Euclidean metric. On calculates

$$g(a,b) + \mathrm{i}g(a,\mathrm{i}b) = h(a,b).$$

Hence h is determined by g. Conversely this formulae defines a Hermitean form for a given real bilinear form if and only if

$$g(a,b) = g(\mathrm{i}a,\mathrm{i}b).$$

It is also interesting to look at the imaginary part of h,

$$h(a,b) = g(a,b) + \mathrm{i}A(a,b).$$

Obviously A is a real bilinear form which is alternating,

$$A(a,b) = -A(a,b).$$

It is closely related to g, one checks easily

$$A(a,b) = g(a,\mathrm{i}b).$$

Hence A determines h and moreover:

A real alternating bilinear form on V defines a Hermitean form if and only if

$$A(a,b) = A(\mathrm{i}a,\mathrm{i}b).$$

We generalize this to complex vector bundles $E \to X$ on a differentiable manifold X. A Hermitian form h on E is a family of Hermitean forms h_a on E_a, which depends differentiable on a. This means that the function

$$h(X,Y) := h_a(X_a, Y_a)$$

is differentiable for any differentiable sections X, Y of E over some open subset of X. The imaginary part A of h can be considered as a section of the bundle with the fibres

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(E_a, \mathbb{R}).$$

5.1 Definition. *A Hermitian manifold (X, h) is a complex analytic manifold together with a Hermitean metric on the tangent bundle TX.* DHm

Recall that the real tangent bundle of a complex analytic manifold carries a complex structure. There are reasons to write the multiplication with i in T_aX by a new letter

$$J : T_aX \longrightarrow T_aX.$$

If (X, h) is a Hermitian manifold, then there is an underlying structure as Riemannian manifold (X, g), $g = \operatorname{Re} h$. But also the imaginary part $\Omega = \operatorname{Im} h$ is interesting. Recall that this is a section of the bundle with the fibres

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T_a(X), \mathbb{R}).$$

This means that Ω is a real alternating differential form of degree 2.

5.2 Definition. *Let (X, h) be a Hermitian manifold. The real differential form* DFf

$$\Omega = \operatorname{Im} h \in A^2_{\mathbb{R}}(X)$$

is called the fundamental form.

It is easy to verify that Ω is of type $(1, 1)$, hence

$$\Omega \in A^2_{\mathbb{R}}(X) \cap A^{1,1}(X).$$

This can be also seen form the following formulae in local coordinates:

Let $U \subset \mathbb{C}^n$ be an open subset. The tangent space can identified with $\mathbb{C}^n$. A Hermitian form h on $\mathbb{C}^n$ is given by the Hermitean Gram matrix

$$h(e_i, e_j),$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{C}^n$. A Hermitian form on the tangent bundle hence is given by a differentiable function $h(x)$ with values in the space of Hermitian $n \times n$-matrices. The fundamental form can be computed:

$$\Omega = \mathrm{i} \sum_{1 \le i,j \le n} h_{ij}\; dz_i \wedge d\bar{z}_j.$$

We omit the (easy) computation.

To introduce the star operator we need an orientation. But complex analytic manifolds are always naturally oriented. This comes form the following simple observation. Let $e_1, \dots, e_n$ be a basis of the complex vector space. Then $e_1, \mathrm{i}e_1, \dots, e_n, \mathrm{i}e_n$ is a basis of the underlying real vector spaces. Basises obtained in this way are orientation compatible. (When $f_1, \dots, f_n$ is a second basis, then one has a complex $n \times n$-transition matrix A and a $2n \times 2n$-transition matrix for the corresponding real basises. One has $\det B = |\det A|^2 > 0$.) We use these real basises to define the orientation of V. We defined the operator $* : A^p_{\mathbb{R}}(X) \to A^{2n-p}_{\mathbb{R}}(X)$. We extend this to a $\mathbb{C}$-linear operator

$$* : A^p(X) \longrightarrow A^{2n-p}(X).$$

Here we denote by n the complex dimension of X which is assumed to be pure dimensional.

5.3 Lemma. *The star operator on a Hermitian manifold of pure dimension n preserves the (p,q)-graduation as follows:* SpG

$$A^{p,q}(X) \longrightarrow A^{n-q,n-p}(X).$$

It satisfies $** = (-1)^{p+q}$.

We now define the **complex codifferentiation** as

$$\bar\partial^* := -\bar * \,\bar\partial\, \bar * : A^{p,q}(X) \longrightarrow A^{p,q-1}(X)$$

It can be checked that for compact X this operator satisfies

$$(\bar\partial\alpha,\beta) = (\alpha,\bar\partial^*\beta) \quad \text{where} \quad (\alpha,\beta) := \int_X \alpha\wedge *\bar\beta.$$

We define the complex Laplace-Beltrami operators as:

$$\bar\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{p,q}(X) \longrightarrow A^{p,q}(X)$$

and similarly

$$\Box = \partial\partial^* + \partial^*\partial.$$

We denote by

$$\mathcal{H}^{p,q}(X) \subset A^{p,q}(X)$$

the kernel of $\bar\Box$.

The point is that $\bar\Box$ is also an elliptic operator. Similar arguments as in the real case show:

5.4 Main theorem of complex Hodge theory. *Let X be a compact Hermitean manifold. Then $\mathcal{H}^{p,q}(X)$ is contained in the space of $\bar\partial$-closed forms and the natural homomorphism* MTH

$$\mathcal{H}^{p,q}(X) \xrightarrow{\sim} H^{p,q}(X)$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called **Hodge-numbers**

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$$

are well defined numbers.

There is also a duality result: Before we formulate it, we introduce a slight generalization of the notion of a degenerated pairing. Let V, W be two finite dimensional complex vector spaces. Consider a $\mathbb{R}$-bilinear map $(\cdot,\cdot) : V\times W \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first variable. Then we get a natural $\mathbb{R}$-linear map $V \to \mathrm{Hom}_{\mathbb{C}}(W,\mathbb{C})$, which sends $a\in V$ to the linear form $x\mapsto (a,x)$. We call the pairing non-degenerated, if this map is an isomorphism. Then $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W$.

5.5 Duality. *Let X be a pure n-dimensional compact Hermitean manifold. The integral $\int_X \alpha \wedge *\bar{\beta}$ induces a non-degenerated pairing* PD

$$\mathcal{H}^{p,q}(X) \times \mathcal{H}^{n-p,n-q}(X) \longrightarrow \mathbb{C},$$

hence

$$h^{p,q}(X) = h^{n-p,n-q}(X).$$

We finally remark that there is also an analogous result for the ∂-complex. One has to replace $\bar{\Box}$ by $\Box$.

6. Hodge theory of holomorphic bundles

Let $E \to X$ be a complex vector bundle over a differentiable manifold. We consider the complex bundle with he fibre

$$\bigwedge^{d} \mathrm{Hom}_{\mathbb{R}}(T_a X, \mathbb{C}) \otimes_{\mathbb{C}} E_a$$

Its differentiable sections are denoted by $A^d(X,E)$. There is now directly available operator $d : A^d(X,E) \to A^{d+1}(X,E)$. Nevertheless reasonable operators exist and are called connections of E. We don't want to treat connections here and discuss a modified situation:

We assume now that X is a complex analytic manifold and that $E \to X$ is a *holomorphic* vector bundle. We consider the bundle with the fibres

$$\bigwedge^{pq} \mathrm{Hom}_{\mathbb{R}}(T_a X, \mathbb{C}) \otimes_{\mathbb{C}} E_a.$$

Its differentiable sections are denoted by

$$A^{p,q}(X,E).$$

We want to show that there is a naive operator

$$\bar{\partial} : A^{p,q}(X,E) \longrightarrow A^{p,q+1}(X,E).$$

The reason will be that

$$\bar{\partial}(f\omega) = f\bar{\partial}$$

for *holomorphic*. This is true because $\partial f = 0$. To make the construction clear, we formulate a lemma, which is rather trivial but nevertheless essential:

Let $U \subset \mathbb{C}^n$ be an open subset and M an module over the ring $\mathcal{O}$ of holomorphic functions. Let

$$\mathcal{C}^\infty(U)^d \times \mathcal{O}(U)^m \longrightarrow M$$

be a map which is $\mathcal{O}(U)$-bilinear. Then there is a unique $\mathcal{O}(U)$-linear map L which maps the following diagram commutative:

$$\begin{array}{ccc} \mathcal{C}^\infty(U)^d \times \mathcal{O}(U)^m & \longrightarrow & W \\ & \searrow & \uparrow L \\ & & \mathcal{C}^\infty(U)^{(d,m)} \end{array}$$

Here $\mathcal{C}^\infty(U)^{(d,m)}$ denotes the space of $d \times m$-matrices with coefficients in $\mathcal{C}^\infty(U)$ and the inclined arrow is defined by $(f, g) \mapsto (f_i g_j)$. In algebraic terms this is the universal property of the tensor product

$$\mathcal{C}^\infty(U)^d \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^m = \mathcal{C}^\infty(U)^{(d,m)},$$

which we didn't introduce. We don't need it here because this universal property is enough for our purpose and trivial as mentioned.

We denote in the following by $E^\infty(U)$ the space of *holomorphic* sections of the holomorphic bundle E. A slight generalization states: Let $U \subset X$ be a sufficiently small open subset (such that it is contained in the domain of definition of a holomorphic chart and such E is trivial over U) and let

$$A^{p,q}(U) \times E^\infty(U) \longrightarrow M \qquad \text{(some } \mathcal{O}(U)\text{-module)}$$

be a $\mathcal{O}(U)$-bilinear map. Then there exists a unique $\mathcal{O}(U)$-linear map such that the following diagram commutes:

$$\begin{array}{ccc} A^{p,q}(U) \times E^\infty(U) & \longrightarrow & M \\ & \searrow & \uparrow L \\ & & A^{p,q}(U, E) \end{array}$$

The inclined arrow is defined fibre wise by means of the tensor product.

We apply this for $M = A^{p,q+1}(U)$ and the map

$$A^{p,q}(U) \times E^\infty(U) \longrightarrow A^{p,q+1}(U, E), \qquad \omega \otimes s \longmapsto (\bar\partial \omega) \otimes s,$$

which is defined by fibre wise tensoring. The fact is that this map is $\mathcal{O}(U)$-linear and the universal property hence gives us the desired map

$$\bar\partial : A^{p,q}(U, E) \longrightarrow A^{p,q+1}(U, E).$$

It is clear that this maps can be glued globally to a map

$$\bar{\partial} : A^{p,q}(X,E) \longrightarrow A^{p,q+1}(X,E).$$

It is clear that $\bar{\partial} \circ \bar{\partial} = 0$. Hence we can define the generalized **Dolbeault-cohomology**

$$H^{p,q}(X,E) := \frac{\operatorname{Kernel}(A^{p,q}(X,E) \longrightarrow A^{p,q+1}(X,E))}{\operatorname{Image}(A^{p,q-1}(X,E) \longrightarrow A^{p,q}(X,E))}.$$

We want to define a Laplace-Beltrami operator in this context. Therefore we have to define an operator

$$\bar{\partial}^* : A^{p,q}(X,E) \longrightarrow A^{p,q-1}(X,E)).$$

In the case of a compact X, this operator should be adjoint to $\bar{\partial}$ with respect to some Hermitian product on $A^{p,q}(X,E)$. Recall that in the case of absence of E this has been defined as integral $\alpha \wedge \bar{\beta}$. Hence we are lead to look for operators

$$A^{p,q}(X,E) \xrightarrow{*} A^{n-q,n-p}(X,E), \quad A^{p,q}(X,E) \times A^{n-p,n-q}(X,E) \xrightarrow{\wedge} \mathcal{C}^\infty(X).$$

The star operator is no problem. In the case where E is absent it was induced by a bundle map and this bundle map can be tensored with E fibre wise. In a similar way we get a pairing

$$A^{p,q}(X,E) \times A^{n-p,n-q}(X,E^*) \xrightarrow{\wedge} \mathcal{C}^\infty(X),$$

where E^* is the dual bundle of E. This pairing also is defined fibrewise in an obvious way. We need a link between E and E^*.

Assumption. *E is equipped with a Hermitian metric.*

When V is a Hermitian vector space with Hermitian form $\langle \cdot, \cdot \rangle$ we get a natural map

$$\sharp : V \longrightarrow V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

It maps an $a \in V$ to the linear form $x \longmapsto \langle x, a \rangle$. This map is an isomorphism of real vector spaces and satisfies $\sharp(Ca) = \bar{C}\sharp(a)$. We call such a map am antilinear map.

Let $A \to B$ and $C \to D$ be two antilinear maps of complex vector spaces, then $A \otimes_{\mathbb{C}} B \to C \otimes_{\mathbb{C}} D$ is a well-defined antilinear map.

Using these general remarks, we get an antilinear map

$$\natural := \bar{*} \otimes \sharp : A^{p,q}(X,E) \longrightarrow A^{n-p,n-q}(X,E^*).$$

Its inverse is $(-1)^{p+q}\bar{*} \otimes \sharp^{-1}$.

Now can define the operator

$$\bar\partial^* := -\natural\,\bar\partial\,\natural : A^{p,q}(X,E) \longrightarrow A^{p,q-1}(X,E)$$

and the generalized Laplace-Beltrami operator

$$\bar\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{p,q}(X,E) \longrightarrow A^{p,q}(X,E).$$

To show the conistence of this definition we introduce for compact X a Hermitian scalar product on $A^{p,q}(X,E)$:

$$\langle\alpha,\beta\rangle = \int_X \alpha\wedge\natural\beta.$$

One checks that this is positive definit and that $\bar\partial^*$ is adjoint to $\bar\partial$.

We denote by $\mathcal{H}^{p,q}(X,E) \subset A^{p,q}(X,E)$ the kernel of $\bar\Box$.

6.1 Hodge theory for holomorphic vector bundles. *Let X be a compact Hermitean manifold and $E \to X$ a holomorphic vector bundle which is equipped with a Hermitian metric. Then $\mathcal{H}^{p,q}(X,E)$ is contained in the space of $\bar\partial$-closed forms and the natural homomorphism* MTH

$$\mathcal{H}^{p,q}(X,E) \xrightarrow{\sim} H^{p,q}(X,E)$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the sp-called **Hodge-numbers**

$$h^{p,q}(X,E) := \dim_{\mathbb{C}} H^{p,q}(X,E)$$

are well defined numbers.

There is also a duality result:

6.2 Serre-duality. *Let X be a pure n-dimensional compact Hermitian manifold and $E \to X$ a holomorphic vector bundle. The integral $\int_X \alpha\wedge\natural\beta$ induces a non-degenerated pairing* PDb

$$\mathcal{H}^{p,q}(X,E)\times\mathcal{H}^{n-q,n-p}(X,E^*) \longrightarrow \mathbb{C},$$

hence

$$h^{p,q}(X,E) = h^{n-p,n-q}(X,E^*).$$

We have to remark, that in the formulation of the Serre-duality no metric of E occurs. We mention that using the technique of partition of unity it is easy to show that every complex bundle admits a Hermitean metric. Hence 6.2 is true in general.

Chapter IV. Sheaves

1. Presheaves

1.1 Definition. *A presheaf $\mathcal{F}$ of sets on a topological space is an assignment to each open subset $U \subset X$ of a set $\mathcal{F}(U)$ and to each pair of open subsets $V \subset U$ of a map* DpG

$$r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

such that

a) $r_U^U = \mathrm{id}_{\mathcal{F}(U)}$,

b) $r_W^V \circ r_V^U = r_W^U \qquad$ *($W \subset V \subset U$ open).*

We call r_V^U the restriction map and use sometimes the notation

$$s|V = s|_{\mathcal{F}}V := r_V^U(s) \qquad (s \in \mathcal{F}(U).$$

The elements of $\mathcal{F}(U)$ are called sections over U. In the case $U = X$ they are called global sections.

We give some examples of presheaves:

1. Let $(X, \mathcal{O}_X)$ be a geometric space. Then $\mathcal{O}_X$ defines a preasheaf.
2. Let $f : E \to X$ be a continuous map of topological spaces. A section over an open subset $U \subset E$ by definition is a map $s : U \to E$ with the property $f(s(x)) = x$ for all $x \in U$. We denote be $\mathcal{F}(U)$ the set of all sections. It is clear how to define the restriction maps to get a presheaf.
3. Let X be a topological space and A some set. We define the constant preashef by $\mathcal{F}(U) = A$, the restriction maps are taken as the identity map.

A *map of presheaves* $f : \mathcal{F} \to \mathcal{G}$ by definition is a family of maps of sets $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with the restriction maps. This means that the diagrams

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

commute for all open $V \subset U$. It is clear how to compose maps of presheafs $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{h} \mathcal{H}$, namely $(h \circ f)_U = h_U \circ f_U$ and it is clear that there is an identity map $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$. A map of presheaves f is an isomorphism if all f_U are bijective. Then the inverse map f^{-1} is defined in the obvious way.

A presheaf $\mathcal{F}$ is called a **subpresheaf** of the presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all U and if the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \subset & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \subset & \mathcal{F}(V) \end{array}$$

commutes for all open $V \subset U$. We write $\mathcal{F} \subset \mathcal{G}$ to express this. The natural inclusions $i_U : \mathcal{F}(U) \to \mathcal{G}(U)$ then define a map of presheaves $i : \mathcal{F} \to \mathcal{G}$, which also is called the natural inclusion.

A map of presheaves $\mathcal{F} \to \mathcal{G}$ is called presheaf-injective, if all $\mathcal{F}(U) \to \mathcal{G}(U)$ are injective. Then this map is the composition of an isomorphim of $\mathcal{F}$ onto a subsheaf of $\mathcal{G}$ and its canonical injection into $\mathcal{G}$.

Let $f : \mathcal{F} \to \mathcal{G}$ be a map of preasheaves. We consider $\mathcal{H}(U) := f_U(\mathcal{F}(U))$. The restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ map $\mathcal{H}(U)$ into $\mathcal{H}(V)$. This means that the family od all $\mathcal{H}(U)$ defines a subpresheaf of $\mathcal{G}$. This preasheaf is denoted by $f(\mathcal{F})$. It is called the **presheaf-image** of f.

Let $f, g : \mathcal{F} \to \mathcal{G}$ be two maps of presheaves. We can consider

$$\mathcal{K}(U) := \big\{\, x \in \mathcal{F}(U); \quad f_U(x) = g_U(x) \,\big\}.$$

This defines a subpresheaf $\mathcal{K}$ of $\mathcal{F}$. The subpresheaf $\mathcal{K}$ is called the **preasheaf-kernel** of the pair (f, g).

Stalks

For a geometric structure we defined $\mathcal{O}_{X,a}$. In the same way we define $\mathcal{F}_a$ for a presheaf $\mathcal{F}$ on a toplogical space and $a \in X$. We consider the set of pairs (U, s) and (V, t), where $a \in U, V \subset X$ are open neighborhoods and $s \in \mathcal{F}(U)$, $t \in \mathcal{G}(V)$. Two such pairs are called equivalent, if there exists an open neighbourhood $a \in W \subset U \cap V$ with the property $s|W = t|W$. The equivalence class of (U, s) is denoted by $[U, s]_a$ and sometimes simply by s_a. The set of all equivalence classes is the so-called stalk $\mathcal{F}_a$. For every open neighborhood $a \in U \subset X$ we have a natural map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_a, \quad s \longmapsto s_a,$$

the notation

$$\mathcal{F}_a = \varinjlim \mathcal{F}(U)$$

is in accordance with the standard definition of a direct limit (which we don't need).

Let $f : \mathcal{F} \to \mathcal{G}$ be a map of preasheaves.For every point a there is an induced map

$$f_a : \mathcal{F}_a \longrightarrow \mathcal{G}_a.$$

It is characterized by the property that the diagrams

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_a & \longrightarrow & \mathcal{G}_a \end{array}$$

commute for all open $a \in U$. If $h : \mathcal{G} \to \mathcal{H}$ is a second map of presheaves then $(h \circ f)_a = h_a \circ f_a$.

Presheaves with additional structure

A presheaf of groups is a presheaf of sets $\mathcal{F}$ with an additional structure: Every $\mathcal{F}(U)$ carries a structure as group and all the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are homomorphims. Then the stalks $\mathcal{F}_a$ also carry structures as groups such that the maps $\mathcal{F}(U) \to \mathcal{F}_a$ are homomorphims. Let $\mathcal{F}, \mathcal{G}$ be two presheaves of groups. A map of presheaves $f : \mathcal{F} \to \mathcal{G}$ is called a homomorphism of presheaves of groups if all f_U are homorphims of groups. Then $f(\mathcal{F})$ is a subpresheaf of groups of $\mathcal{G}$, a notion which is defined in the obvious way. A homomorphism of presheaves of groups induces homomorphims of groups in the stalks. The kernel $f : \mathcal{F} \to \mathcal{G}$ of a homomorphism of presheaves of groups is the subpresheaf of groups $\mathcal{K}$ defined by $\mathcal{K}(U) = \text{kernel}(f_U)$. This can be considered as the kernel of the arrow pair (f, g) where g denotes the trivial homomorphism which maps everything to the unit element.

In the same way we define presheaves of rings. In this context we only consider commutative rings and assume that they have a unit element. Homomorphims of rings are assumed to carry the unit element into the unit element. For example a geometric structure is a preasheaf of rings.

Let $\mathcal{O}$ a presheaf of rings. A $\mathcal{O}$-pre-module $\mathcal{M}$ is a presheaf of sets with an additional structure: Every $\mathcal{M}(U)$ carries a structure as $\mathcal{O}(U)$ module. The restrictions $\mathcal{M}(U) \to \mathcal{M}(V)$ are assumed to be homomorphims of abelian groups with the property

$$(fs)|V = (f|V)(s|V) \qquad \big(V \subset U,\ f \in \mathcal{O}(U),\ s \in \mathcal{M}(U)\big).$$

Then the stalks $\mathcal{M}_a$ carry natural structures of $\mathcal{O}_a$-modules. A map of presheaves $f : \mathcal{M} \to \mathcal{N}$ between to $\mathcal{O}$-modules is called $\mathcal{O}$-linear, if $\mathcal{M}(U) \to$

$\mathcal{N}(U)$ is $\mathcal{O}(U)$-linear for all open $U \subset X$. Then $\mathcal{M}_a \to \mathcal{N}_a$ is $\mathcal{O}_a$-linear for all $a \in X$. There are some other constructions, which we mention briefly:

1. The composition of $\mathcal{O}$-linear maps is $\mathcal{O}$-linear, the identity is $\mathcal{O}$-linear, there is a natural notion of $\mathcal{O}$-linear isomorphism.
2. There is an obvious notion of a $\mathcal{O}$-pre-submodule $\mathcal{N} \subset \mathcal{M}$. The definition is made such the the natural inclusion is $\mathcal{O}$-linear.
3. The presheaf-image resp. the presheaf-kernel of a $\mathcal{O}$-linear map $f : \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-pre-moduls is a $\mathcal{O}$ pre-submodule of $\mathcal{M}$ resp. $\mathcal{N}$.
4. Let $\mathcal{N} \subset \mathcal{M}$ be a $\mathcal{O}$-pre-submodule. Then every $\mathcal{M}(U)/\mathcal{N}(U)$ is a $\mathcal{O}(U)$-module. There is a unique way to define restriction maps $\mathcal{M}(U)/\mathcal{N}(U) \to \mathcal{M}(V)/\mathcal{N}(V)$ such that we obtain an $\mathcal{O}$-module $\mathcal{F}/\mathcal{G}$ and such that the natural projections $\mathcal{M}(U) \to \mathcal{M}(U)/\mathcal{N}(U)$ define a $\mathcal{O}(U)$-linear map. We call $\mathcal{M}/\mathcal{N}$ the factor-pre-module.

The following example is basic in our context:

Let $\pi : E \to X$ be a holomorphic vector bundle over the complex analytic manifold $(X, \mathcal{O}_X)$. We denote for an open subset $U \subset X$ by $E^{\text{hol}}(U)$ the set of all holomorphic sections over U, i.e. holomorphic mappings $s : U \to E$ with the property $\pi(s(x)) = x$ for $x \in U$. Obviously $E^{\text{hol}}(U)$ carries a structure as $\mathcal{O}_X(U)$-module. There are obvious restriction maps such that $\mathcal{E}$ gets a structure as $\mathcal{O}_X$-pre-module. Similar constructions are possible in the real analytic and differentiable world.

We will later see (5.1) that E^{hol} contains all information of E. Hence the notion of presheaves can be considered as a generalization of the notion of a vector bundle.

2. Sheaves

2.1 Definiton. *A presheaf $\mathcal{F}$ is called a sheaf, if the following assumptions are satisfied:* DGe

G1 *Let $U = \bigcup U_i$ be an open covering of an open subset of X and let $s, t \in \mathcal{F}(U)$ be two sections with the property*

$$s|U_i = t|U_i \quad \textit{for all} \quad i.$$

Then $s = t$.

G2 *Let $s_i \in \mathcal{F}(U_i)$ be a family of sections with the property*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j \quad \textit{for all} \quad i, j.$$

Then there exists $s \in \mathcal{F}(U)$ with $s|U_i = s_i$ for all i.

G3 *$\mathcal{F}(\emptyset)$ consists of one element.*

In this context we want to mention that we follow the convention that there is precisely one map from the empty set into some arbitary set, namely the empty map.

Let $\mathcal{F},\mathcal{G}$ be sheaves. A map of presheaves $f:\mathcal{F}\to\mathcal{G}$ is also called a map of sheaves. A sub-sheaf of a sheaf is a sub-presheaf which is also a sheaf. Let $f,g:\mathcal{F}\to\mathcal{G}$ be a pair of maps of sheaves. It is easy to see that the kernel is a sheaf. We hence call it the **sheaf-kernel**.

There is a problem with the image of a map of sheaves $f:\mathcal{F}\to\mathcal{G}$. Usually the presheaf-image will not be a sheaf. We give an example: Let X be the complex plane, $\mathcal{O}$ the sheaf of holomorphic functions and $\mathcal{O}^*$ the subsheaf consisting of all holomorphic functions without zeros. There is a map of sheaves

$$\exp:\mathcal{O}\longrightarrow\mathcal{O}^*,\quad \exp(f)=e^f.$$

The presheaf-image $\mathcal{F}$ contains all holomorphic functions which admit a holomorphic logarithm. This presheaf is no sheaf. Consider for example $U=\mathbb{C}^*$ and $f(z)=1/z$. It is known that f admits a holomorphic logarithm in any disc which is contained in U. It is also known that there is no holomorphic logarithm on the whole $\mathbb{C}^*$. We can cover U by discs and obtain that the assumption G2 for sheaves is violated.

The generated sheaf

We will remedy the above problem by means of a construction which attaches to each presheaf $\mathcal{F}$ a sheaf $\hat{\mathcal{F}}$ which differs as little as possible from $\mathcal{F}$. For this purpose we introduce first a certain monster sheaf which sometimes called the associated flasque sheaf.

2.2 Definition and Remark. *Let $\mathcal{F}$ be a presheaf. We define* DaW

$$\mathcal{F}^{(0)}(U):=\prod_{a\in U}\mathcal{F}_a\qquad (U\ \textit{open}).$$

If one takes natural projections as restriction maps, then $\mathcal{F}^{(0)}$ is a sheaf.

This sheaf is of course very huge, because there is no binding between the members $s_a\in\mathcal{F}$ of a family $(s_a)\in\mathcal{F}^{(0)}$. We will now introduce such a binding and get then a much more reasonable sheaf. Firstly we notice that there is a natural map of presheaves

$$\mathcal{F}(U)\longrightarrow\mathcal{F}^{(0)}(U),\quad s\longmapsto(s_a)_{a\in U}\qquad (s_a:=[U,s]_a).$$

2.3 Remark. *The natural $\mathcal{F}\to\mathcal{F}^{(0)}$ map is presheaf-injective if $\mathcal{F}$ satisfies* G1 *and* G3. iIS

Proof. Let $s,t\in\mathcal{F}(U)$ two elements with the same image. Then the germs are equal and, if $U\neq\emptyset$, we obtain that there exists an open covering of U by subsets U_i such that $s|U_i=t|U_i$. Then G1 implies $s=t$. □

2.4 Definiton and Remark. *Let $\mathcal{F}$ be a presheaf. Let U be an open subset. An system $S \in \mathcal{F}^{(0)}(U)$ is called* ***coherent****, if any $a \in U$ admits an open neighborhood $a \in U(a) \subset U$ such that $S|U(a)$ is in the presheaf-image of the natural map $\mathcal{F} \to \mathcal{F}^{(0)}$. We denote by $\hat{\mathcal{F}}(U)$ the set of all coherent systems. This defines a subsheaf $\hat{\mathcal{F}}$ of $\mathcal{F}^{(0)}$. There is a natural map* DeG

$$\mathcal{F} \longrightarrow \hat{\mathcal{F}},$$

which we call canonical.

The sheaf $\hat{\mathcal{F}}$ should be considered as the best possible approximation of $\mathcal{F}$ by a sheaf. This can be made mathematically precise:

2.5 Proposition (Universal property of the generated sheaf). *Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ a sheaf. For every map of $f : \mathcal{F} \to \mathcal{G}$ there exists a unique map $\hat{f} : \hat{\mathcal{F}} \to \mathcal{G}$, such that f is the composition of $\hat{f}$ with the canonical map $\mathcal{F} \to \hat{\mathcal{F}}$.* UeG

Proof. Let U be an open subset and $S = (S_a)$ an element of $\hat{\mathcal{F}}(U)$. By definition there exists an open covering $U = \bigcup U_i$ and elements $(s_i \in \mathcal{F}(U_i)$ such that $S_a = (s_i)_a$ for all $a \in U_i$ and all i. The images $t_i = f(s_i) \in \mathcal{G}(U_i)$ fit together in the sense that $(t_i)_a = (t_j)_a$ for all $a \in U_i \cap U_j$. It follows from G1 that $t_i|U_i \cap U_j = t_j|U_i \cap U_j$ and then that they glue to an element $t \in \mathcal{G}(U)$. We have to define $t = \hat{f}(S)$. □

Let $f : \mathcal{F} \to \mathcal{G}$ be a map of presheaves. We can apply 2.5 to the composition $\mathcal{F} \to \mathcal{G} \to \hat{\mathcal{G}}$ and obtain a canonical map $\hat{f} : \hat{\mathcal{F}} \to \hat{\mathcal{G}}$. If $h : \mathcal{G} \to \mathcal{H}$ is another map of presheaves then $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.

We collect some simple facts about the generated sheaf and omit the simple proofs.

Is $\mathcal{F}$ is a subpresheaf of a sheaf $\mathcal{G}$. Then the inclusion $\mathcal{F} \to \mathcal{G}$ induces a map $\hat{\mathcal{F}} \to \mathcal{G}$ which is presheaf-injective. Hence $\hat{\mathcal{F}}$ is mapped isomorphically to a subsheaf of $\mathcal{G}$. This subsheaf can be concretely described as follows. An element $s \in \mathcal{G}(U)$ belongs to this subsheaf, if there is an open covering of U by open subsets such that $s|U_i \in \mathcal{F}(U_i)$ for all i.

2.6 Definition. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. We denote by* DpI

$$\hat{f}(\mathcal{F}) := \widehat{f(\mathcal{F})}$$

the sheaf generated by the presheaf-image an consider it is subsheaf of $\mathcal{G}$.

2.7 Remark. *Let $\mathcal{F}$ be a presheaf. The maps* HnW

$$\mathcal{F}_a \longrightarrow \hat{\mathcal{F}}_a \qquad (a \in X)$$

induced by the canonical map are bijective.

If $\mathcal{F}$ is a presheaf of groups or rings, then $\hat{\mathcal{F}}$ carries also a natural structure as sheaf of groups or rings. The universal property 2.5 and 2.7 have obvious variants.

Let $\mathcal{O}$ be a sheaf of rings. By a $\mathcal{O}$-module we understand an $\mathcal{O}$-pre-module which is also a sheaf. Let $\mathcal{O}$ be a presheaf of rings and $\mathcal{M}$ an $\mathcal{O}$-premodule. Then $\hat{\mathcal{M}}$ carries a natural structure as $\hat{\mathcal{O}}$-module and again there are obvious variants of 2.5 and 2.7.

Let $\mathcal{O}$ be a sheaf of rings, $\mathcal{N}$ an $\mathcal{O}$-module and $\mathcal{M}$ an $\mathcal{O}$-submodule. We introduced the factor-pre-sheaf $\mathcal{N}/\mathcal{M}$. This is general not a sheaf. Hence we consider the generated sheaf

$$\widehat{\mathcal{N}/\mathcal{M}}$$

and call this the **factor-module** of $\mathcal{M}$ by $\mathcal{N}$.

Let A be group or ring. Let A_X denote the constant presheaf $A_X(U) = A$. This is usually no sheaf. There is an explicit construction of the generated sheaf $\hat{A}_X$: Let $\mathcal{F}(U)$ be the set of all locally constant functions $U \to R$. This obviousely is sheaf of groups or rings and there is natural homomorphism of sheafs $A_X \to \mathcal{F}$. The homomorphism $\hat{A}_X \to \mathcal{F}$ which comes from the universal property is an isomorphism. Usually we identify $\mathcal{F}$ and $\hat{A}_X$.

We recall that every abelian group can be considered as a $\mathbb{Z}$-module. Hence every presheaf of abelian groups can by considered as $\mathbb{Z}_X$-module. As a consequence every sheaf of abelian groups can be considered als a $\hat{\mathbb{Z}}_X$-module. Hence the theory of $\mathcal{O}$-modules covers the theory of sheaves of abelian groups.

3. Exact sequences

Let $f : A \to B$ be a R-linear map of R-modules. Sometimes we use the notation

$$B/A := B/f(A).$$

A sequence of R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact if the kernel of g equals the image of f. Merely from the condition kernel$(f) \subset$ image(g) follows that there is an obvious linear map $B/A \to C$. The exactness means that this homomorphim induces an isomorphism

$$B/A \xrightarrow{\sim} g(B).$$

A sequence $0 \to A \to B$ is exact if and only $A \to B$ is injective. A sequence $A \to B \to 0$ is exact if and only $A \to B$ is surjective.

A sequence

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

is called exact at i if the image of $A_{i-1} \longrightarrow A_i$ equals the kernel of $A_i \longrightarrow A_{i+1}$. It is called exact, if it is exact at every place. A sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if $A \to B$ is injective and if the induced map $B/A \to C$ is an isomorphism. We call such an sequence a short exact sequence. The typical short exact sequence is

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0,$$

where $B \subset A$ is a sub-module and the arrows are the natural ones.

Exact sequences of preasheaves

The notion of exact sequences of R-modules carries over in an obvious and trivial manner to presehaves. Let $\mathcal{O}$ be a preasheaf of rings on a topological space X.

3.1 Definition. *A sequence $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$ of $\mathcal{O}$-modules is exact in the sense of presheaves if and only if the kernel of g equals the presheaf image of f.* DeP

Nearly trivial is:

3.2 Remark. *A sequence* ReP

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P}$$

*of $\mathcal{O}$-premodules is **exact in the sense of presheaves** if*

$$\mathcal{M}(U) \longrightarrow \mathcal{N}(U) \longrightarrow \mathcal{P}(U)$$

is exact for all open U.

We denote by 0 the zero $\mathcal{O}$-module. It assigns the module 0 to all open sets. If $\mathcal{M}$ is an arbitray $\mathcal{O}$-module there exists exactly one $\mathcal{O}$-linear map from 0 to $\mathcal{M}$ and from $\mathcal{M}$ to 0.

3.3 Remark. *A sequence $0 \to \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-pre-modules is exact in the sense of presheaves if and only if $\mathcal{M}(U) \to \mathcal{N}(U)$ is injective for all open U. A sequence $\mathcal{M} \to \mathcal{N} \to 0$ is exact in the sense of presheaves if and only if $\mathcal{M}(U) \to \mathcal{N}(U) \to 0$ is surjective for all open U. A sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is exact if and only if the induced homomorphism $\mathcal{N}/\mathcal{M} \to \mathcal{P}$ is an isomorphism. Here $\mathcal{N}/\mathcal{M}$ denotes the quotient pre-module.* ReE

There is an additional remark which turns out to be important.

3.4 Remark. *Let* ExH

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P}$$

be an exact sequence the sense of presheaves. Then

$$\mathcal{M}_a \longrightarrow \mathcal{N}_a \longrightarrow \mathcal{P}_a$$

is exact for all a. (We consider here the $\hat{\mathcal{O}}$-module $\hat{\mathcal{M}}$ as $\mathcal{O}$-module in an obvious way.

It is all-important that the converse of 3.4 is false. Is $\mathcal{M}$ a pre-module over $\mathcal{O}$, then the sequence $0 \to \mathcal{M} \to \hat{\mathcal{M}} \to 0$ is exact only if $\mathcal{M}$ is a sheaf, but $0 \to \mathcal{M}_a \to \hat{\mathcal{M}}_a \to 0$ is exact by 2.7.

Exact sequences of sheaves.

Let $\mathcal{O}$ now be a sheaf of rings on a topological space. The correct definition of exactness in the sense of sheaves is:

3.5 Definition. *A sequence of $\mathcal{O}$-modules* DeG

$$\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$$

is exact in the sense of sheaves, if the kernel of g equals the sheaf-image of f.

Recall that the sheaf-kernel and the pre-sheaf kernel of g are the same. But the sheaf-image and presheaf-image of f may be different. In some sense "sheaf-exactness" is easier to check than "presheaf-exactness":

3.6 Proposition. *A sequence* CeH

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P}$$

of $\mathcal{O}$-modules is sheaf-exact if and only

$$\mathcal{M}_a \longrightarrow \mathcal{N}_a \longrightarrow \mathcal{P}_a$$

is exact for all points a.

The analogue of 3.3 in the setting of sheaves is

3.7 Proposition. *A sequence $0 \to \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules is exact in the sense of sheaves if and only if it is exact in the sense of presheaves which means that $\mathcal{M}(U) \to \mathcal{N}(U)$ is injective for all open U. This is also equivalent to the fact that $\mathcal{M}_a \to \mathcal{N}_a$ is injective for all points a.* PGe

A sequence $\mathcal{M} \to \mathcal{N} \to 0$ is exact in the sense of sheaves if and only if the sheaf-image of $\mathcal{M}$ is $\mathcal{N}$, or equivalently that $\mathcal{M}_a \to \mathcal{N}_a$ is surjective for all points a.

A sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *is exact in the sense of sheaves, if* $\mathcal{M} \to \mathcal{N}$ *is injective and if there exists a unique isomorphism*

$$\widehat{\mathcal{N}/\mathcal{M}} \xrightarrow{\sim} \mathcal{P}$$

such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \widehat{\mathcal{N}/\mathcal{M}} & \longrightarrow 0 \\ & & \| & & \| & & \downarrow & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} & \longrightarrow 0 \end{array}$$

commutes.

We know that the last statement is equivalent to the exactness of $0 \to \mathcal{M}_a \to \mathcal{N}_a \to \mathcal{P}_a$ for all points. This sequence implies

$$\widehat{\mathcal{N}/\mathcal{M}}_a \cong \mathcal{N}_a/\mathcal{M}_a \qquad \text{(natural isomorphism)}$$

for all points a.

We conclude this section with one of the most important short exact sequences:

3.8 Remark. *Let X be a complex analytic manifold. The following sequence of sheaves of abelian groups* FEs

$$0 \longrightarrow \hat{\mathbb{Z}}_X \longrightarrow \mathcal{O} \stackrel{f \to \exp(2\pi i f)}{\longrightarrow} \mathcal{O}^* \longrightarrow 1$$

is exact.

Here 1 means the trivial sheaf of groups similar to 0 now written multiplicatively.

4. Flabby sheaves

We consider in this section sheaves of abelian groups on a topological space (which can be considered as $\hat{\mathbb{Z}}_X$-modules). Because we are interested mainly in sheaves and preseheaves only are auxiliary, we make the following changes in the notation:

1. An exact sequence of sheaves always means a sequence of sheaves which is *exact in the sense of sheaves.*
2. If $f : \mathcal{F} \to \mathcal{G}$ is a map of sheaves we denote in contrast to our earlier notation by $f(\mathcal{F})$ the sheaf-image (and not the presheaf-image).

3. If $\mathcal{G} \subset \mathcal{F}$ is a subsheaf (of sheaves of abelian groups or more general $\mathcal{O}$-modules. then we denote by $\mathcal{F}/\mathcal{G}$ the factor-sheaf and not the factor-presheaf. Recall that the factor-sheaf is the sheaf generated by the factor preasheaf $\mathcal{F}(U)/\mathcal{G}(U)$.
4. We use the notations

$$\Gamma\mathcal{F} := \Gamma(X, \mathcal{F}) := \mathcal{F}(X).$$

4.1 Lemma. *Let* KkK

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \to 0$$

be an exact sequence of sheaves of abelian groups, then

$$0 \longrightarrow \Gamma\mathcal{F} \longrightarrow \Gamma\mathcal{G} \longrightarrow \Gamma\mathcal{H}$$

is exact.

The last map needs not to be surjective. Cohomology is a method to measure this absence of surjectivity in the sense that one constructs a long exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow & \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \\ & H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \dots . \end{aligned}$$

Hence $H^0(X, \mathcal{F}) := \mathcal{F}(X)$ is just the beginning of an infinite sequence of groups. For the construction we use Godements approach using flabby sheaves.

4.2 Definition. *A sheaf $\mathcal{F}$ is called flabby (german: welke Garbe; french: faiscaux flasque), if the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ all are surjective.* DfS

For example the sheaf $\mathcal{F}^{(0)}$ from 2.2 is a flabby sheaf. This example shows that any sheaf is a subsheaf of a flabby sheaf.

The class of flabby sheaves has two basic properties:

4.3 Lemma. *Let* EwG

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of sheaves. Then the following two properties hold:

1. *If $\mathcal{F}$ is flabby, then*

$$0 \longrightarrow \Gamma\mathcal{F} \longrightarrow \Gamma\mathcal{G} \longrightarrow \Gamma\mathcal{H} \longrightarrow 0$$

is exact.

2. *If $\mathcal{F}$ and $\mathcal{G}$ are flabby, then $\mathcal{H}$ is flabby.*

Proof of 1. Let $\bar{s} \in \Gamma\mathcal{H}$. We have to construct an $s \in \mathcal{G}(X)$, which maps to $\bar{s}$. For the proof we consider the set of all pairs

$$(U, s),\ U \subset X \text{ open},\ s \in \mathcal{G}(U); \quad s \text{ maps to } \bar{s}|U.$$

This set can be ordered:

$$(U, s) > (U', s') \iff U \subset U' \text{ and } s|U' = s'.$$

From Zorn's lemma follows that there exists a maximal (U, s). We have to show that $U = X$. We argue indirect and assume $U \neq X$. We choose a point $a \in X$. Because $\mathcal{G}_a \to \mathcal{H}_a$ is surjective there exists an open neighborhood V of a and a section $s' \in \mathcal{G}(V)$, which maps to $\bar{s}|V$. The section $s'|U \cap V - s|U \cap V$ maps to zero and is hence contained in $\mathcal{F}(U \cap V)$. Since $\mathcal{F}$ is flabby this section is the restriction of a section $t \in \mathcal{F}(V)$. We can replace s' by $s' - t$ and can therefore assume

$$s'|U \cap V = s|U \cap V.$$

From G2 follows that there exits an $s_0 \in \mathcal{G}(U \cup V)$ with $s_0|U = s$ and $s_0|V = t$. Then $(U \cup V, s_0) > (U, s)$ in contradiction to the maximality of (U, s).

Proof of 2. in 4.3. We use a trivial construction for sheaves: Let $\mathcal{F}$ be a presheaf on X and $U \subset X$ an open subset. Then the presheaf $\mathcal{F}|U$ on U is defined in the trivial way: $\mathcal{F}|U(V) = \mathcal{F}(V)$. If $\mathcal{F}$ is a flabby sheaf then $\mathcal{F}|U$ is flabby too. Let now U be open in X and $s \in \mathcal{H}(U)$. We have to show that s extends to a global section. We use now the already proved first part for U instead of X. Because $\mathcal{F}|U$ is flabby, there exists an inverse image $t \in \mathcal{G}(U)$ of s. Because $\mathcal{G}$ is flabby, t is the restriction of a $t_0 \in \Gamma\mathcal{G}$. We consider the image $s_0 \in \Gamma\mathcal{H}$ of t_0. This section extends s. □

As a consequence of 4.3 we prove:

4.4 Lemma. *Let* WeW

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow \cdots$$

be an exact sequence of flabby sheaves (indexed by natural numbers). Then

$$0 \longrightarrow \Gamma\mathcal{F}_1 \longrightarrow \Gamma\mathcal{F}_2 \longrightarrow \Gamma\mathcal{F}_3 \longrightarrow \cdots$$

is exact.

Proof. We set $\mathcal{F}_0 = 0$. First we claim that the shaves

$$\mathcal{F}_{m+1}/\mathcal{F}_m := \mathcal{F}_{m+1}/\mathrm{Bild}(\mathcal{F}_m) \qquad (m \geq 0)$$

all are flabby. For this we observe that the short sequences

$$0 \longrightarrow \mathcal{F}_{m+1}/\mathcal{F}_m \longrightarrow \mathcal{F}_{m+2} \longrightarrow \mathcal{F}_{m+2}/\mathcal{F}_{m+1} \longrightarrow 0$$

are exact. The claim follows now by induction from the second part of 4.3. From the first part now follows the exactness of

$$0 \longrightarrow \Gamma(\mathcal{F}_{m+1}/\mathcal{F}_m) \longrightarrow \Gamma\mathcal{F}_{m+2} \longrightarrow \Gamma(\mathcal{F}_{m+2}/\mathcal{F}_{m+1}) \longrightarrow 0.$$

It is easy to derive from this fact 4.4. One has to use that $\Gamma\mathcal{F}_{m+1} \to \Gamma(\mathcal{F}_{m+1}/\mathcal{F}_m)$ is surjective (use again 4.4). □

Notice: During the proof we used twice a technique, which reduces the statement of exactness of a sequence to the exactness of many short exact sequences. We will use this technique later too.

We introduced the "canonical flabby sheaf" $\mathcal{F}^{(0)}$ associated to a sheaf $\mathcal{F}$. We define now inductive an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \longrightarrow \mathcal{F}^{(2)} \longrightarrow \cdots,$$

namely

$$\mathcal{F}^{(1)} := (\mathcal{F}^{(0)}/\mathcal{F}), \quad \mathcal{F}^{(2)} := (\mathcal{F}^{(1)}/\mathcal{F})^{(0)}, \quad \mathcal{F}^{(3)} := (\mathcal{F}^{(2)}/\mathcal{F})^{(1)}, \ldots.$$

This long exact sequence is called the *canonical flabby resolution* of $\mathcal{F}$ or also the *Godement-resolution* of $\mathcal{F}$.

We collect the basic properties of this resolution.

4.5 Proposition. *Let $\mathcal{F}$ be a sheaf of abelian groups. The Godement resolution is a canonical exact sequence* kGF

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \longrightarrow \mathcal{F}^{(2)} \longrightarrow \cdots,$$

where all the $\mathcal{F}^{(m)}$ are flabby. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups. There exist natural homomorphims (vertical arrows) such that the diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}^{(0)} & \longrightarrow & \mathcal{F}^{(1)} & \longrightarrow & \mathcal{F}^{(2)} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}^{(0)} & \longrightarrow & \mathcal{G}^{(1)} & \longrightarrow & \mathcal{G}^{(2)} & \longrightarrow \cdots \end{array}$$

commute. The vertical arrows ar functioral, which means:

a) *In the case $\mathcal{G} = \mathcal{F}$ and the identity map $\mathcal{F} \to \mathcal{G}$ the vertical arrows are all identity arrows.*
b) *The vertical arrows are compatible with composition of homomorphims $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in an obvious sense.*

If $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves then all sequences

$$\mathcal{F}^{(m)} \to \mathcal{G}^{(m)} \to \mathcal{H}^{(m)} \to 0$$

are exact too.

5. Vector bundles and sheaves

Let $\mathcal{O}$ be a sheaf of rings (commutative and with unit) over X. We need some simple constructions for $\mathcal{O}$-modules, for example the cartesian product $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ is defined in an obvious way. An $\mathcal{O}$-module is called free, if it is isomorphic to the $\mathcal{O}$-module $\mathcal{O}^n = \mathcal{O} \times \cdots \times \mathcal{O}$. The number n is uniquely determined and called the rank. A module and **locally free**, if every point admits an open neighbourhood U such that $\mathcal{M}|U$ is a free $\mathcal{O}|U$-module. Then the rank can be considered as locally constant function on X. if it is constant we call it the rank of the module. Let's give an example:

Let $E \to X$ be a (real or complex) differentiable vector bundle over a differentiable manifold. If one assigns to an open subset U the set of differentiable sections $E^\infty(U)$ one obtains a sheaf, more precisely a module of the sheaf $\mathcal{C}_X^\infty$ of (real or) complex differentiable functions. This module is locally free. It is quite clear that a differentiable bundle map $E \to F$ induces a map $E^\infty \to F^\infty$ of $\mathcal{C}^\infty$-modules.

5.1 Proposition. *Let X be a differentiable manifold. Every locally free $\mathcal{C}_X^\infty$-module is isomorphic to a module E^∞, where E is a differentiable vector bundle. If E, F are two differentiable vector bundles, then there is a one-to-one correspondence between differentiable bundle maps $E \to F$ and $\mathcal{C}_X^\infty$-linear maps of sheaves $E^\infty \to F^\infty$. Especially the two bundles E, F are isomorphic (as differentiable vector bundles) if and only the corresponding $\mathcal{C}_X^\infty$-modules are isomorphic.* VgG

Additional remark. *There is a holomorphic version for complex analytic manifolds X: Every holomorphic vector bundle E induces a locally free $\mathcal{O}_X$-module E^{hol}. Holomorphic vector bundles correspond to locally free $\mathcal{O}_X$-modules.*

We should mention that II.5.4 was a special case of this and II.5.5 a variant. We leave it to the reader to a generalization of 5.1 which includes also multilinear bundle maps $E_1 \times \cdots \times E_n \to F$. They correspond to $\mathcal{C}_X^\infty$-multilinear maps $E_1^\infty \times \cdots \times E_n^\infty \to F^\infty$.

We mention that there may be important maps of sheaves $E^\infty \to F^\infty$, which are linear over the ground field $\mathbb{R}$ or $\mathbb{C}$ but which are not $\mathcal{C}^\infty$-linear. Such maps then cannot be induced by bundle maps. An important example for this is the exterior derivative $d : A^m(U) \to A^{m+1}(U)$ which is not $\mathcal{C}^\infty(U)$ linear. Instead of this one has a product rule. Hence the exterior derivative is not induced by a bundle map.

We give an example of a directly constructed locally free line bundle: Let $(X, \mathcal{O}_X)$ be a geometric space and $A \subset X$ a subset. We consider for open $U \subset X$ the set

$$\mathcal{J}(U) := \{\, f \in \mathcal{O}_X(U); \quad f|(A \cap U) = 0 \,\}.$$

This is a subsheaf of $\mathcal{O}_X$, more precisely an $\mathcal{O}_X$ submodule of $\mathcal{O}_X$. Because $\mathcal{J}(U)$ is an ideal in $\mathcal{O}_X(U)$, we call $\mathcal{J}$ the ideal sheaf associated to A. We are interested in cases where this submodule is free:

5.2 Remark and notation. *Let $Y \subset X$ be a (holomorphically) smooth closed subset of the complex analytic manifold $(X, \mathcal{O}_X)$ of pure codimension one ($\dim_a Y = \dim_a X - 1$ for $a \in Y$). The ideal sheaf $\mathcal{J}$ associated to Y is locally free of rank one, i.e. it corresponds to a holomorphic line bundle L.* UvL

We denote this sheaf by $\mathcal{J}_Y$.

We know that $\mathcal{J}_Y$ must be isomorphic to the sheaf associated to a line bundle L_Y. This line bundle can be defined directly by means of transition functions:

6. Subspaces and sheaves

We are interested in the following situation. Let $(X, \mathcal{O}_X)$ be a geometric space (in our application a complex analytic variety) and $Y \subset X$ a closed subset. Recall that we defined the restricted geometric structure $\mathcal{O}_Y = \mathcal{O}_X|Y$. We are interested in $\mathcal{O}_X$-modules $\mathcal{M}$ with the property $\mathcal{M}|(X - Y) = 0$. We want to show that such sheaves correspond to sheaves on Y. For this we defined the restriction $\mathcal{M}|Y$. (This restriction could be defined in a much more general context, here we defined it only for $\mathcal{O}_X$ and for $\mathcal{O}_X$-modules, which vanish outside Y). To define $\mathcal{N} := \mathcal{M}|Y$ we have to consider an open subset $V \subset X$. This is the intersection $V = U \cap Y$. We want to define

$$\mathcal{N}(V) = \mathcal{M}(U).$$

The problem is of course the uniqueness of this definition. Let U' be another open subset of X with $V = U' \cap Y$. Using the fact that Y is closed in X and that $\mathcal{M}$ vanishes outside Y is it easy to see, that the restriction maps

$$\mathcal{M}(U), \mathcal{M}(U') \xrightarrow{\sim} \mathcal{M}(U \cap U')$$

are isomorphisms. This shows the claimed independence. To get a logical clean definition one should better define

$$(\mathcal{M}|Y)(V) := \varinjlim \mathcal{M}(U).$$

Clearly $\mathcal{M}|Y$ carries a structure as $\mathcal{O}_Y$-module.

There is a converse construction (which also has vast generalizations): Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Denote by $j : Y \to X$ the canonical injection. We define a sheaf $j_*\mathcal{N}$ on X by the assignment

$$(j_*\mathcal{N})(U) := \mathcal{N}(U \cap Y).$$

This is an $\mathcal{O}_X$ module. It vanishes outside Y, hence $j_*(\mathcal{N}|Y)$ is defined.

6.1 Proposition. *Let $(X, \mathcal{O}_X)$ be a geometric space and $j : Y \hookrightarrow X$ the natural inclusion of a closed subset. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module which vanishes outside Y. There is a natural isomorphismus* DuI

$$\mathcal{M} \xrightarrow{\sim} j_*(\mathcal{M}|Y).$$

Moreover there are natural isomorphisms of the stalks

$$\mathcal{M}_a \xrightarrow{\sim} (\mathcal{M}|Y)_a \qquad (a \in Y).$$

Let $\mathcal{N}$ by an $\mathcal{O}_Y$ module. There is a natural isomorphism

$$\mathcal{N} \xrightarrow{\sim} (j_*\mathcal{N})|Y.$$

6.2 Remark. *Let X be a complex analytic manifold and Y a smooth subset of pure codimension one, $j : Y \hookrightarrow X$ the natural injection. Let $\mathcal{J}_Y$ the ideal sheaf of Y. There is a natural exact sequence of $\mathcal{O}_X$-modules* KeI

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_Y \longrightarrow 0.$$

There are various generalizations: Let M be a module over a ring R (commutative and with unit). Let $\mathfrak{a} \subset R$ be an ideal then $\mathfrak{a}M$ is defined as the set of all finite sums of elements of the form am, $a \in \mathfrak{a}$, $m \in M$. This is a submodule of M. Let more generally $\mathcal{O}$ be a sheaf of rings and $\mathcal{M}$ an $\mathcal{O}$-module and $\mathcal{J} \subset \mathcal{M}$ an ideal sheaf. Then $U \mapsto \mathcal{J}(U)\mathcal{M}(U)$ is a presheaf. The generated sheaf is denoted by $\mathcal{J}\mathcal{M}$. This is is a $\mathcal{O}$-submodule of $\mathcal{M}$. It is easy to prove that there is a natural isomorphism

$$\mathcal{J}_a\mathcal{M}_a \xrightarrow{\sim} (\mathcal{J}\mathcal{M})_a.$$

In other words, the sequence

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/(\mathcal{J}\mathcal{M}) \longrightarrow 0$$

is exact. A generalization of 6.2 states:

6.3 Remark. *Let $\mathcal{O}$ be a sheaf of rings (commutative and with unit), $\mathcal{J} \subset \mathcal{O}$ and ideal sheaf, $\mathcal{M}$ and $\mathcal{O}$-module. Assume that $Y \subset X$ is a closed subset such that $\mathcal{J}|(X - Y) = \mathcal{O}|(X - Y)$. Then there is a natural exact sequence* veS

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_*((\mathcal{M}/\mathcal{J}\mathcal{M})|Y) \longrightarrow 0.$$

Assume furthermore that there exists an open neighborhood $Y \subset U \subset X$ such that $\mathcal{M}|U \cong \mathcal{O}|U$. Then the sequence can be written as

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_*((\mathcal{O}/\mathcal{J})|Y) \longrightarrow 0.$$

Assume furthermore, that $\mathcal{O} = \mathcal{O}_X$ is a geometric structure and that $\mathcal{J} = \mathcal{J}_Y$ is its ideal sheaf. Then the sequence can be written as

$$0 \longrightarrow \mathcal{J}_Y\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_*(\mathcal{O}_X|Y) \longrightarrow 0.$$

Finally we give an criterion that an $\mathcal{O}_X$-module is locally free of

Chapter V. Cohomology

1. Some homological algebra

1.1 Definition. *A complex $A^\bullet$ of abelian groups is a sequence of homomorphims of abelian groups indexed by $\mathbb{Z}$* DK

$$\cdots \longrightarrow A^n \stackrel{d}{\longrightarrow} A^{n+1} \stackrel{d}{\longrightarrow} A^{n+2} \longrightarrow \cdots$$

such that

$$d \circ d = 0,$$

equivalently

$$\text{image}(A^n \longrightarrow A^{n+1}) \subset \text{kernel}(A^{n+1} \longrightarrow A^{n+2}).$$

A homomorphism of complexes $\varphi^\bullet : A^\bullet \to B^\bullet$ is a sequence of homomorphims $\varphi^n : A^n \to B^n$ such that all diagrams

$$\begin{array}{ccc} A^n & \longrightarrow & A^{n+1} \\ \downarrow & & \downarrow \\ B^n & \longrightarrow & B^{n+1} \end{array}$$

commute.

Correct logical notation would demand that we give d an indices like $d^n_{A^\bullet}$. But for better lookout we try to avoid to use indices as long as confusions can be excluded. Even more, we will avoid using letters for maps as long it is clear from the context, which map is considered at the moment.

There is an obvious composition of homomorphisms of complexes $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$.

For an complex $A^\bullet$ we consider

$$C^n(A^\bullet) := \text{kernel}(A^n \longrightarrow A^{n+1}), \quad B^n(A^\bullet) := \text{image}(A^{n-1} \longrightarrow A^n).$$

The condition $d \circ d = 0$ means

$$B^n(A^\bullet) \subset C^n(A^\bullet).$$

1.2 Definition. *The cohomology groups of a complex $A^\bullet$ are* DKk

$$H^n(A^\bullet) := C^n(A^\bullet)/B^n(A^\bullet) \qquad (n \in \mathbb{Z}).$$

A complex is an exact sequence if and only of all cohomology groups vanishes. Cohomology should be considered as a measure for the deviation from exactness.

1.3 Remark. *A complex homomorphism $A^\bullet \to B^\bullet$ induces natural homomorphism* CHi

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups. They are compatible with composition of complex homomorphims.

It should be clear, how these homomorphisms have to be defined. □

All the constructions of this section up to now can be repeated for R-modules and for sheafs of abelian groups on a topolgical space ore more generally for $\mathcal{O}$-modules, where $\mathcal{O}$ is a fixed sheaf of rings. We describe now a construction, which works well for abelian groups and R-modules and which we don't need for sheaves.

The combining homomorphism

We consider an exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0.$$

Here 0 denotes of course the zero complex which containes only zero groups. Exactness means that all sequences

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0$$

are exact sequences of abelian groups. We will construct for each n a not quite obvious homomorphism of groups

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Construction of δ*.* Let $c \in H^n(C^\bullet)$. Chose a representative in C^n and of this representative a preimage $b \in B^n$. We consider $db \in B^{n+1}$. From $d(db)$ and from the exactness of $0 \to A^{n+1} \to B^{n+1} \to C^{n+1} \to 0$ follows that db is the image of an $a \in A^{n+1}$. Clearly $da = 0$. Hence a represents an element of $H^{n+1}(A^\bullet)$. It is easy to check that this element is independent of the choices we made during the construction. We call this element by $\delta(c)$. This defines the so-called **combining homomorphism**

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

This construction is one of the most fundamental constructions of homological algebra. The basic properties are:

1.4 Proposition. *Let* $0 \to A^\bullet \to B^\bullet \to C^\bullet$ *be an exact sequence of complexes. The long sequence* LeS

$$\cdots \longrightarrow H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \longrightarrow \cdots$$

is exact. Moreover this long sequence is natural in the following sense: Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow 0 \\ & & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow 0 \end{array}$$

be a commutative diagram of complexes with exact limes. The diagram

$$\begin{array}{ccc} H^n(C^\bullet) & \xrightarrow{\delta} & H^{n+1}(A^\bullet) \\ \downarrow & & \downarrow \\ H^n(C'^\bullet) & \xrightarrow{\delta} & H^{n+1}(A'^\bullet) \end{array}$$

is commutative.

The proof is a straight forward diagram chasing, which we omit. □

We need another lemma of homological algebra:

1.5 Lemma. *Let* LkA

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A_{00} & \longrightarrow & A_{01} & \longrightarrow & A_{02} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A_{10} & \longrightarrow & A_{11} & \longrightarrow & A_{12} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A_{20} & \longrightarrow & A_{21} & \longrightarrow & A_{22} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & \vdots & & \vdots & & \vdots & \end{array}$$

be a commutative diagram such that the rows and columns are complexes. We assume that every column is exact besides the first one. We consider the colomology of the first column

$$H^q(0 \longrightarrow A_{00} \longrightarrow A_{10} \longrightarrow \cdots) = \frac{\text{Kernel}(A_{q+1,1} \longrightarrow A_{q+2,1})}{\text{Image}(A_{q,1} \longrightarrow A_{q+1,1})}$$

and similarly of the first row. There is a natural homomorphism

$$H^q(0 \longrightarrow A_{11} \longrightarrow A_{21} \longrightarrow \cdots) \longrightarrow H^q(0 \longrightarrow A_{11} \longrightarrow A_{12} \longrightarrow \cdots), \quad q \geq 1.$$

All these homomorphism are isomorphisms if also all the lines besides the first one are exact.

Additional Remark. *If one only knows that*

$$0 \longrightarrow A_{20} \longrightarrow A_{21} \quad \textit{and} \quad 0 \longrightarrow A_{10} \longrightarrow A_{11} \longrightarrow A_{12}$$

are exact then the homorphism in the case $q = 1$ is injective.

Scetch of proof. We construct the map in the first cohomology. Consider a cohomology class α represented by $a \in A_{20}$. The image in A_{30} and then in A_{31} is 0. Denote the image of a in A_{21} by b. Because the image of b in A_{31} is zero and we are in an exact column, there is an $c \in A_{11}$ which maps to b. We consider the image $d \in A_{12}$ of c. This maps to 0 in A_{12} because this image comes form a. Since we are in an exact column we get $e \in A_{02}$ which maps to d. The image of e in A_{03} is 0 because A_{03} is embedded in A_{13} and d maps to 0 in A_{13}. Hence we have defined a cohomology class β in the first cohomolgy of the first row. It easy to check that this cohomology clase is independent of the choices we made.

2. Sheaf-cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups on the topological space Using the Godement resolution we associate to $\mathcal{F}$ the following complex $\mathcal{F}^\bullet$:

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \longrightarrow \cdots,$$

where $\mathcal{F}^{(0)}$ sits at the zero-position of the complex. This complex is exact outside the zero position. We apply Γ and obtain a complex $\Gamma\mathcal{F}^\bullet$.

$$\cdots \longrightarrow 0 \longrightarrow \Gamma\mathcal{F}^{(0)} \longrightarrow \Gamma\mathcal{F}^{(1)} \longrightarrow \cdots,$$

which needs not to be exact at positive positions. We define

$$H^n(X, \mathcal{F}) := H^n(\Gamma\mathcal{F}^\bullet)$$

and call this groups the cohomology groups of $\mathcal{F}$. They are zero of course for negative n. Another exceptional case is $n = 0$. Here

$$H^0(X,\mathcal{F}) = \mathrm{kernel}(\mathcal{F}^{(0)}(X) \longrightarrow \mathcal{F}^{(1)}(X)) \cong \mathcal{F}(X).$$

This is a canonical isomorphism. Hence we may write

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X).$$

We collect the basic properties of this construction:

2.1 Theorem. *Let X be a topological space. There is an assignment for any sheaf of abelian groups $\mathcal{F}$ to a sequence $H^n(X,\mathcal{F})$ of abelian groups with the following properties:* LeS

1. *One has*

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X), \quad H^n(X,\mathcal{F}) = 0 \text{ for } n < 0.$$

2. *If $\mathcal{F}$ is flabby, then $H^n(X,\mathcal{F}) = 0$ for $n > 0$.*
3. *Every homomorphism of sheaves $\mathcal{F} \to \mathcal{G}$ induces homomorphims*

$$H^n(X,\mathcal{F}) \to H^n(X,\mathcal{F}),$$

which are natural in the following sense. They are compatible with composition and the identity map induces identity maps. Especially an isomorphism induces isomorphims.

2. *For every short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ there exists (by an explicit construction) a long exact sequence*

$$\cdots \longrightarrow 0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H}) \longrightarrow H^1(X,\mathcal{F})$$
$$\longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H}) \longrightarrow H^2(X,\mathcal{F}) \longrightarrow \cdots$$

Additional remarks. *If $\mathcal{F}$ is a sheaf of R-modules then the cohomology groups are R-modules too. If one restricts to sheaves of R-modules and R-linear maps of sheaves of R-modules then all occurring arrows above are R-linear.*

If $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ an $\mathcal{O}$-module, then $\mathcal{F}$ can be considered as sheaf of R-modules, where $R = \Gamma(X,\mathcal{O})$ and $\mathcal{O}$-linear maps of $\mathcal{O}$-modules are R-linear too.

Example. Let X be a connected complex analytic manifold. We introduced the short exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

In contrast to an earlier notation we denote by $\mathbb{Z}_X$ here not the constant presheaf but the sheaf generated by it, which can be interpreted as the sheaf of locally constant functions with values in $\mathbb{Z}$. Because X is connected, we have $\Gamma(X, \mathbb{Z}_X) \cong \mathbb{Z}$. We will use the notation

$$H^n(X, \mathbb{Z}) := H^n(X, \mathbb{Z}_X)$$

(and similarly for other abelien groups instead of $\mathbb{Z}$. From the long exact cohomology sequence we obtain

$$0 \to \mathbb{Z} \longrightarrow \Gamma(X, \mathbb{Z}_X) \longrightarrow \Gamma(X, \mathcal{O}) \longrightarrow \Gamma(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathbb{Z}).$$

Thus the elements of $H^1(X, \mathbb{Z})$ can be obstructions for an holomorphic function $f : X \to \mathbb{C}$ without zeros to have a holomorphic logarithm. This is a problem of complex analysis. One should notice in this context that $H^n(X, \mathbb{Z})$ only depends on the topology of X and not on the special complex analytic structure.

We see that it can be helpful to compute cohomology groups explicitely. Unfornately the way as we defined them is not very useful for explicit calculations because the Godement complex is terribly big. We will develop no methods which are better for explicit calculations.

3. Soft and fine sheaves

3.1 Definition. *A sheaf $\mathcal{F}$ of abelian groups on a topological space is called acyclic, if* DAa

$$H^n(X, \mathcal{F}) = 0 \quad \textit{for} \quad n > 0.$$

An acyclic resolution of a sheaf $\mathcal{F}$ of abelian groups is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \cdots,$$

where all $\mathcal{F}_n$ are acyclic.

We know that flabbay sheaves are acyclic and that the Godement resolution is an acyclic resolution.

3.2 Lemma. *Let* BKa

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \cdots$$

be an acylic resolution. Then

$$H^q(X, \mathcal{F}) \cong \text{kernel}(\Gamma\mathcal{F}_q \to \Gamma\mathcal{F}_{q+1})/\text{image}(\Gamma\mathcal{F}_{q-1} \to \Gamma\mathcal{F}_q).$$

We will construct bigger classes of acyclic sheaves to obtain resolutions which are more down to earth than the Godement resolution.

3.3 Definition. *A class $\mathcal{K}$ of sheaves of abelian groups on a topologocal space is called a **Godement class**, if the following holds.* DGK

1. *Every sheaf is isomorphic to a subsheaf of an acyclic element of $\mathcal{K}$.*
2. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be an exact sequence:*
   a) $\mathcal{F}, \mathcal{G} \in \mathcal{K} \Longrightarrow \mathcal{H} \in \mathcal{K}$.
   b) *If $\mathcal{F} \in \mathcal{K}$ then*
$$0 \longrightarrow \Gamma\mathcal{F} \longrightarrow \Gamma\mathcal{G} \longrightarrow \Gamma\mathcal{H} \longrightarrow 0$$

is exact.

We know that the flabby sheaves perform a Godement class.

3.4 Remark. *All members of a Godement class are acyclic.* MgA

Proof. This shows the same proof as that of IV.4.4.

We defined
$$\mathcal{F}^{(0)}(Y) := \prod_{y \in Y} \mathcal{F}_y$$

for open subsets $Y \subset X$. This definition is possible for arbitrary subsets. Also the notion of a coherent system is possible. A system $S := (s_y)_{y \in Y}$ is called coherent, if every point $a \in Y$ admits an open neighbourhood $U \subset X$ (open in X) and a section $t \in \mathcal{F}(U)$ such that $[U, t]_y = s_y$ for all $y \in Y$. We recall that for open $Y \subset X$ the map

$$\mathcal{F}(Y) \longrightarrow \{\text{coherent systems on } U\}$$

is bijective (because we assume that $\mathcal{F}$ is a sheaf. Hence it is justified to *define* for an arbitrary subset Y

$$\mathcal{F}(Y) := \{\text{coherent systems on } U\}.$$

If $Z \subset Y$ is a subset, one has a natural restriction map

$$\mathcal{F}(Y) \longrightarrow \mathcal{F}(Z).$$

This means, that we extended $\mathcal{F}$ to a presheaf on X, equipped with the discrete topology. By the way
$$F(\{y\} = F_y.$$

3.5 Definition. *A sheaf $\mathcal{F}$ on a topological space is called **soft**, if* DSg

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$$

*is surjective for every **closed subset** $Y \subset X$.*

The notion of a soft sheaf is very useful for paracompact spaces X. We recall from general topology this notion and its basic properties:

*An open covering $X = \bigcup U_i$ is called **locally finite**, if every point of X admits a neighborhood W such that only for finitely many indices i the intersection $W \cap \bar{U}_i$ is not empty.*

*An open covering $X = \bigcup_{j \in J} V_j$ is called a **refinement** of the open covering $X = \bigcup_{j \in I} U_I$ (different index-sets are allowed, if there exists for every $i \in I$ an index $j = j(i) \in V_j$ such that*

$$V_{j(i)} \subset U_i.$$

*A topological space is called **paracompact**, if it is Hausdorff and if every open covering admits a locally finite refinement.*

From general topology one knows:

Every metric space is paracompact.
Every locally compact Hausdorff space with countable basis of the topology is paracompact.

"Locally compact" means that every point admits a compact neighborhood. Hence our differentiable and real and complex analytic manifolds are a paracompact (because we made as a general assumption that they have countable basis of the topology. By the way we notice that every closed subset (equipped with the induced topology) of a paracompact space is paracompact.

The basic properties of paracompact spaces are:

Let $X = \bigcup U_i$ a locally finite covering of a paracompact space: Then there exist:

1. *An open subset $V_i \subset U_i$ with the properties*

$$\bar{V}_i \subset U_i \quad \textit{and} \quad X = \bigcup V_i.$$

2. *A continous function $f_i : X \to \mathbb{R}$, $0 \le f_i \le 1$ such that*

$$f_i(x) \neq 0 \Longrightarrow x \in V_i, \quad \textit{and} \quad \sum_i f_i(x) = 1 \textit{ for all } x.$$

If X is a differentiable manifold, the functions f_i can be taken as differentiable functions.

Notice that the sum is essentially finite.

After these topological preperations we come to the basic properties of soft sheaves:

3.6 Porposition. *Let X be paracompact. For every closed subset $Y \subset X$ and every $s \in \mathcal{F}(Y)$ there exists an open subset $U \subset X$ with $Y \subset U$ such that s in the image of* SiL

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(Y).$$

Corollary. *Every flabby sheaf is soft.*

Proof. Consider $(s_a)_{a\in Y} \in \mathcal{F}(Y)$. There exists an open covering $Y \subset \bigcup U_i$ and sections $s_i \in \mathcal{F}(U_i)$ sich that $(s_i)_a = s_a$ for all $a \in Y \cap U_i$. We may assume that this covering is locally finite. (Add to the covering $X - Y$ to get a covering of X and apply the paracompactness of X.) We consider the following set

$$W := \{\, x \in \bigcup U_i; \quad x \in \bar{V}_i \cap \bar{V}_j \Longrightarrow (s_i)_x = (s_j)_x \,\}.$$

Clearly W contains Y. We are ready if we can show that there exists an open set $Y \subset U \subset W$, becausy by G2 the sections s_i will glue in U to a common section. This means that we have to construct for every $x \in Y$ a neighborhood $W(x)$ (neighborhood in X) with $W(x) \subset W$. Shrinking several times we find step by step a neighborhood which satisfies:

a) The set J of indices i sich that $W(x)$ meets $\bar{V}_i$ is finite.
b) $x \in \bigcup_{j\in J} \bar{V}_j$. (We can $W(x)$ intersect with the complements of the finitely many $\bar{V}_j$, which don't contain x.
c) $W(x) \subset \bigcap U_{j\in J} U_j$.
d) The $s_i|W(x)$ are all the same.

Obviously $W(x)$subsetW □

3.7 Proposition. *The class of soft sheaves on a paracompact space is a Godement class, hence soft sheaves are acylcic.* EPg

3.8 Definition. *A sheaf of abelian groups $\mathcal{F}$ on a paracompact space X is called* ***fine****, if for every locally finite covering $(U_i)_{i\in I}$ there exists a system of sheaf homomorphisms* DFg

$$l_i : \mathcal{F} \longrightarrow \mathcal{F} \qquad (i \in I)$$

with the following property:

For $a \notin U_i$ the induced homomorphism of the stalk $(s_i)_a : \mathcal{F}_a \to \mathcal{F}_a$ is zero and

$$\sum_{i\in I} (s_i)_a = 0. \qquad \textit{(finite sum).}$$

3.9 Lemma. *Fine sheaves are soft, hence acyclic* FsS

3.10 Proposition. *Let $\mathcal{O}$ by the sheaf of continuos functions on a paracompact space or the sheaf of differentiable functions on a differentiable manifold. Every $\mathcal{O}$-module is fine, hence acyclic,* Cpa

4. The theorem of de Rham

Let X be a differentiable manifold of dimension n. We consider the sheaf $\mathcal{A}_X^p$ of real-valued alternating differentiable forms on X. They are members of the sequence

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{A}_X^0 \longrightarrow \mathcal{A}_X^1 \longrightarrow \cdots \longrightarrow \mathcal{A}_X^n \longrightarrow 0.$$

The lemma of Poincarè tells us that this sequence is exact. This is a fine resolution of $\mathbb{R}_X$ and we obtain:

4.1 Theorem of de Rham. *There are isomorphisms (given by explicit constructions)* TDr

$$H^q(X,\mathbb{R}) \cong H^q_{\mathrm{dR}}(X,\mathbb{R}) = \frac{\text{closed differential forms of degree } q \text{ on } X}{\text{total differential forms of degree } q \text{ on } X},$$

especially $H^q(X,\mathbb{R}) = 0$ *for* $q > n$.

In the usual formulations of de-Rham‘s theorem $H^q(X,\mathbb{R})$ is defined as singular cohomology group in the sense of algebraic topology. For sake of completeness we explain briefly without detailed proofs the link between singular and sheaf-cohomology.

The standard simpley Δ_n is the convex hull of 0 and the unit-vectors in $\mathbb{R}^{n+1}$. Let X be a topological space and A an abelian group. We denote by $S^n(X)$ the set of all continuous maps $\Delta^n \to X$ (so-called singular simplices) and by $CS^n(X,A)$ the group of all maps $S^n(X) \to A$ (so-called singular cochains). These groups belong to a complex

$$0 \longrightarrow CS^1(X,A) \xrightarrow{d} CS^1(X,A) \xrightarrow{d} CS^3(X,A) \xrightarrow{d} \cdots$$

with certain standard operators d. The q-th cohomology group of this complex is called the q-the singular cohomology group of X with values in A. We denote ist by $H^q_{sing}(X,A)$. Under good circumstances these singular cohomology groups can be identified with the sheaf cohomolgy groups $H^q(X,A)$. In this context we call a topological space “good” if it is a locally compact Hausdorff space, if it is locally contractible. We don’t give the definition for “locally contractible”, we only mention that manifolds have this property.

4.2 Theorem. *For a good topological space one has a constructive isomorphism* ScG

$$H^q(X,A) \cong H^q_{sing}(X,A).$$

Scetch of proof. the idea is to sheafify $CS^n(X, A)$. In a first step one observes that there is a natural presheaf

$$U \longmapsto CS^n(U, A).$$

This needs not to be a sheaf, but we can consider the generated sheaf

$$\mathcal{CS}^n(X, A).$$

We get a complex of sheaves

$$0 \longrightarrow A_X = \mathcal{CS}^0(X, A) \longrightarrow \mathcal{CS}^1(X, A) \longrightarrow \cdots$$

next one has to prove two properties:
1) The sheaves $\mathcal{CS}^n(X, A)$ are flabby.
2) *The natural map $CS^n(X, A) \to \Gamma\mathcal{CS}^n(X, A)$ is surjective.*
We omit the (short) proofs. From 1) follows that the sheaf cohomology $H^q(X, A)$ equals the cohomology of the complex

$$\cdots 0 \longrightarrow \Gamma\mathcal{CS}^1(X, A) \longrightarrow \mathcal{CS}^2(X, A) \longrightarrow \cdots$$

From Two follows that this equals the cohomology of

$$0 \longrightarrow CS^1(X, A) \xrightarrow{d} CS^2(X, A) \longrightarrow \cdots,$$

which is the singular cohomology. □

5. Čech cohomology

We describe a complete different approach to sheaf cohomology:

5.1 Definition. *Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of a topological space and let F be a sheaf of abelian groups on X. A q-cochain f of $\mathcal{F}$ with respect to $\mathfrak{U}$ is an assignment of any q-tuple* DqC

$$\sigma := (i_0, i_1, \dots, i_q) \in I^{q+1}$$

to an element

$$f(\sigma) = f(i_0, i_1, \dots, i_q) \in \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q}).$$

The q-chaines form an abelian group which we denote by

$$C^q(\mathfrak{U}, \mathcal{F}).$$

For $q < 0$ this is understood to be zero. We define an operator

$$d : C^q(\mathfrak{U}, \mathcal{F}) \longrightarrow C^{q+1}(\mathfrak{U}, \mathcal{F})$$

by

$$df(i_0, i_1, \ldots, i_{q+1}) := \sum_{\nu=0}^{q+1} (-1)^\nu f(i_0, i_1, \ldots \hat{i}_\nu, \ldots, i_{q+1}) \mid U_{i_0} \cap \ldots \cap U_{i_{q+1}}.$$

The above index "hat" means cancellation. It is easy to check $d \circ d = 0$. hence we get a complex, the so-called *Čech complex of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$*.

5.2 Definition. *The* ***Čech cohomology*** DCc

$$H^q(\mathfrak{U}, \mathcal{F})$$

of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$ is the cohomology of the Čech complex.

The Çech cohomolgy vanishes of course for negative q.

5.3 Lemma. *There is a natural isomorphism* nCc

$$H^0(\mathfrak{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F}).$$

Proof. A 0-cochain is family of sections

$$s_i \in \mathcal{F}(U_i).$$

The chain is in the kernel of d if

$$s_i | U_i \cap U_j = s_j | U_i \cap U_j.$$

These cocycles glue to a global section. □

5.4 Lemma. *If $\mathcal{F}$ is flabby, then* wCv

$$H^q(\mathfrak{U}, \mathcal{F}) = 0 \quad \textit{for} \quad q > 0.$$

We come to a connection between the introduced sheaf cohomology and Čech cohomolgy. We consider an acyclic resolution of a sheaf of abelian groups for example the Godement resolution

$$0 \to \mathcal{F} \longrightarrow \mathcal{F}_{/} \longrightarrow \mathcal{F}_{\infty} \longrightarrow \cdots$$

We consider also an open covering $\mathfrak{U}$ of X. There is an associated double complex.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{F}_0) & \longrightarrow & \Gamma(X,\mathcal{F}_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0(\mathfrak{U},\mathcal{F}) & \longrightarrow & C^0(\mathfrak{U},\mathcal{F}_0) & \longrightarrow & C^0(\mathfrak{U},\mathcal{F}_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathfrak{U},\mathcal{F}) & \longrightarrow & C^1(\mathfrak{U},\mathcal{F}_0) & \longrightarrow & C^1(\mathfrak{U},\mathcal{F}_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^2(\mathfrak{U},\mathcal{F}) & \longrightarrow & C^2(\mathfrak{U},\mathcal{F}_0) & \longrightarrow & C^2(\mathfrak{U},\mathcal{F}_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

The columns besides the first one are exact. We obtain a natural homomorphism

$$H^q(\mathfrak{U},\mathcal{F}) \longrightarrow H^q(X,\mathcal{F})$$

which is of course independent of the choice of the resolution..

5.5 Definition. *A covering $\mathfrak{U}$ is called acyclic with respect to the sheaf $\mathcal{F}$, if for every tuple $(i_0,\dots,i_q)$ of indices the restriction* DUa

$$\mathcal{F}|U_{i_0} \cap \ldots \cap U_{i_q}$$

is an acyclic sheaf on $U_{i_0} \cap \ldots \cap U_{i_q}$.

5.6 Theorem of Leray. *Let $\mathfrak{U}$ be an acyclic covering for the sheaf $\mathcal{F}$. There are constructive isomorphisms* TvL

$$H^q(X,\mathcal{F}) \cong H^q(\mathfrak{U},\mathcal{F})$$

for all q.

Additional Remark. *Even in the case of a non-acyclic resolution the natural homomorphism*

$$H^1(\mathfrak{U},\mathcal{F}) \longrightarrow H^1(X,\mathcal{F})$$

is injective.

We introduced already the notion of a refinement $gotV = (V_j)_{j\in J}$ of an open covering $gotU = (U_i)_{i\in I}$. There has to exist a map

$$\tau : J \longrightarrow I \quad \text{with} \quad V_j \subset U_{\tau(j)}.$$

The map τ needs not to be unique. If a refinement map has been chosen, we obtain an obvious homomorphsim $H^q(\mathfrak{U},\mathcal{F}) \longrightarrow H^q(\mathfrak{V},\mathcal{F})$.

5.7 Proposition. *The homomorphism* CVg

$$H^q(\mathfrak{U},\mathcal{F}) \longrightarrow H^q(\mathfrak{V},\mathcal{F})$$

is independent of the refinement map and moreover the diagram

$$\begin{array}{ccc} H^q(\mathfrak{U},\mathcal{F}) & \longrightarrow & H^q(\mathfrak{V},\mathcal{F}) \\ & \searrow \quad \swarrow & \\ & H^q(X,\mathcal{F}) & \end{array}$$

is commutative. Let X be paracompact. For every $a \in H^q(X,\mathcal{F})$ there exists a covering $\mathfrak{U}$ such that a is contained in the image of $H^q(\mathfrak{U},\mathcal{F})$. In the case $q = 1$ paracompactness is not needed.

6. Complex line bundles and their Chern classes

We consider complex line bundle over a topological space. We are interested in the set of all isomorphy classes of such line bundles. (We will see that this is really a set). We denote by $[L]$ the isomorphy class of L. We have two operations on this set. The first one comes from the tensor product, and the second one form dualizing

$$[L_1] + [L_2] := [L_1 \otimes L_2], \quad [L]^{-1} = [L^*].$$

This addition of classes is commutative and associative because of corresponding properties of the tensor product and there is a zero element coming form the trivial bundle $X\times\mathbb{C} \to \mathbb{C}$. Use the obvious canonical isomorphism $V\otimes_{\mathbb{C}}\mathbb{C} \cong V$ for a complex vector space. Using the canonical homomorphism $V \times V^* \cong \mathbb{C}$ for a one-dimensional vector space, we see that the set of isomorphy classes actually is an abelian group, the inverse given $-[L] = [L^*]$. We want to attach to an isomorphy class $[L]$ an element of $H^1(X,\mathcal{C}_X^*)$, where $\mathcal{C}_X^*)$ denotes the sheaf of invertible complex valued continuous functions. For this we consider an open covering $X = \bigcup U_i$ with local trivalizations $h_i : L_{U_i} \to \mathrm{U}_i \times \mathbb{C}$. Recall that this induces an ismorphism

$$h_j \circ h_i^{-1} : U_i \cap U_j \times \mathbb{C} \longrightarrow U_i \cap U_j \times \mathbb{C}.$$

This is in every fibre multiplicattion with a non zero complex number. Hence we obtain an element $h_{ij} \in \mathcal{C}^*(U_i \cap U_j)$. It is clear that we get a Čech cocycle. This cocycle defines an element of $H^1(X, \mathcal{C}_X^*)$. It is easy to check that this cohomology class is independent of the choice of the covering (use a joint refinement of two given coverings). And more over it is clear that isomorphic bundles lead to the same class. Moreover every Čech cocycle is the cocycle of a line bundle by II.1.5. Because H^1 is exhausted by the Čech cohomology groups we obtain:

6.1 Proposition. *The group of isomorphy classes of complex line bundles is canonically isomorphic to the group $H^1(X, \mathcal{C}_X^*)$.* GgH

Variants. *The groups of isomorphy classes of differentiable line bundles over a differentiable manifold is isomorphic to $H^1(X, (\mathcal{C}_X^\infty)^*)$.*

The groups of isomorphy classes of holomorphic line bundles over a complex analytic manifold is isomorphic to $H^1(X, \mathcal{O}_X^*)$.

Recall that we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{C}_X \longrightarrow \mathcal{C}_X^* \longrightarrow 0$$

There is a combining homomorphism

$$\delta : H^1(X, \mathcal{C}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

6.2 Definition. *The **Chern class*** DcC

$$c(L) \in H^2(X, \mathbb{Z})$$

of a complex line bundle is the image of its cohomology class under the combining homomorphism $H^1(X, \mathcal{C}_X^*) \longrightarrow H^2(X, \mathbb{Z})$.

When L is differentiable or holomorphic than one can replace $\mathcal{C}_X$ by $\mathcal{C}_X^\infty$ or $\mathcal{O}_X$ to obtain the same $c(L)$.

We now assume that X is a complex analytic manifold and that L is a holomorphic line bundle. Moreover we assume that L has been equipped with an Hermitian metric. We consider the image $j(c(L))$ under the map

$$j : H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C}).$$

This image must be representable by a differential form $A^2(X)$ via the de Rham-isomorphism. We are going to define such a differential form explicitly.

This form but not its class will depend on the choice of the Hermitian metric. Let $\alpha : L_U \to U \times \mathbb{C}$ be a (holomorphic) local trivialization. The induced Hermitian metric on $U \times \mathbb{C}$ is a function h which attaches every point $a \in U$ to a Hermitian form on $\mathbb{C}$. But a Hermitian form on $\mathbb{C}$ is simply of the form $tz\bar{z}$ with a positive number. Hence h can be considered as (differentiable) function $h_U : U \to \mathbb{R}_{>0}$. Assume that there is another local trivialization $\beta : L_V \to V \times \mathbb{C}$ with the transition function γ which comes form $\beta \circ \alpha^{-1}$. One has $h_V = |\gamma|^2 h_U$ on the intersection. The point now is that the function $\log|\gamma|^2 = \log \gamma\bar{\gamma}$ is annihilated by the operator $\partial\bar{\partial}$. This means that the differential forms $\partial\bar{\partial}\log h_U$ and $\partial\bar{\partial}\log h_V$ agree on the intersection $U \cap V$. Hence they glue to a global differential form, which we denote

$$\partial\bar{\partial}\log h \in A^{1,1}(X).$$

This differential form is annihilated by ∂ and by $\bar{\partial}$, hence also by d. Hence it defines via the de-Rham isomorphism cohomology class in $H^2(X, \mathbb{C})$.

6.3 Proposition. *Let $L \to X$ be a holomorphic line bundle, equipped with a Hermitian metric h. Then* CkL

$$j(c(L)) = \frac{1}{2\pi \mathrm{i}}\partial\bar{\partial}\log h \qquad (\in H^2(X, \mathbb{C})).$$

As a consequence we obtain

$$j(c(L)) \in j(H^2(X, \mathbb{Z})) \cap H^{1,1}(\mathbb{R}).$$

There is a converse result:

6.4 Theorem. *Let X be a complex analytic manifold and $\alpha \in A^{1,1}(X)$ be a real closed differential form whoso class is in the image of $H^2(X, \mathbb{Z})$. Then there exists a holomorphic line bundle (L, h) sich that* Ccc

$$\alpha = \frac{1}{2\pi \mathrm{i}}\partial\bar{\partial}\log h.$$

7. The cohomology of the complex projective space

The projective space carries the *tautological line bundle*. One also can associate to a linear embedded $P^{n-1}(\mathbb{C}) \hookrightarrow P^n(\mathbb{C})$ (induced by a linear injective map $\mathbb{C}^n \to \mathbb{C}^{n+1}$) a line bundle.

7.1 Lemma. *Let $H \subset P^n(\mathbb{C})$ be a linearly embedded $P^{n-1}(\mathbb{C})$. The line bundle associated to H is isomorphic to the inverse (=dual) of the tautological bundle.* tBb

There is another important line bundle K, whose sections are the holomorphic differential forms of highest degree n. This bundle is called the *canonical line bundle.*.

7.2 Lemma. *The canonical bundle on $P^n(\mathbb{C})$ is isomorphic to the n-the power of the tautological bundle.* tBc

It is no problem to compute the Chern classes of these bundles. For this we construct a Hermitian on the tautological line bundle. We take a fixed positive definit Hermitian form on $\mathbb{C}^{n+1}$. To be concrete we take $\sum \bar{z}_i w_i$. We restrict this form to arbitrary lines and obtain a (differentiable) metric h on the tautological line bundle. It is easy to compute the associated 2-form Ω and to verify that the corresponding Hermitian form on that tangent spaces is positiv definit. We obtain:

7.3 Proposition. *The projective space $P^n(\mathbb{C})$ is a Khler manifold such that the Kähler class $\Omega \in H^{1,1}(X)$ is the Chern class of the tautological line bundle. Especially the Kähler class Ω is integral, i.e. contained in the image of $H^2(P^n(\mathbb{C}), \mathbb{Z})$.* psK

This is all what we need in the following. For sake of completeness we describe the complete picture without proof (which is easy if one uses a little algebraic topology):

7.4 Theorem. *The natural map* CPR

$$j : H^m(P^n(\mathbb{C}), \mathbb{Z}) \longrightarrow H^m(P^n(\mathbb{C}), \mathbb{R})$$

is injective, the image generates $H^m(P^n(\mathbb{C}), \mathbb{R})$ as vector space. The space $H^m(P^n(\mathbb{C}), \mathbb{Z})$ is zero for odd m (and for $m > 2n$ and $m < 0$). Moreover

$$H^m(P^n(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}, \qquad m \text{ even and } 0 \le m < 2n$$

A generating element is $\Omega^m = \Omega \wedge \ldots \wedge \Omega$.

Chapter VI. Kaehler manifolds

1. Effective forms

This section contains some non-standard linear algebra. We consider a complex vector space T of dimension n which is equipped with a positive definit Hermitian form

$$h(A,B) = \langle A,B\rangle.$$

We consider the space $\operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C})$, which is a complex vector space of complex dimension $2n$. Recall that there is a decomposition

$$\operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C}) \oplus \overline{\operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C})}$$

into two complex sub-vector spaces. We consider the Grassmann algebra of this complex vector space. Recall that the above decomposition generalizes to

$$A^m := \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C}) = \bigoplus_{p+q=m} A^{p,q}.$$

where $A^{p,q}$ is generated by elements of the form

$$a_1 \wedge \ldots, \wedge a_p \wedge b_1 \wedge \ldots \wedge p_q, \qquad a_i \in \operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C}) \quad b_j \in \overline{\operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C})}.$$

To be more precise: There is an isomorphism

$$\bigwedge^p \operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C}) \otimes_{\mathbb{C}} \bigwedge^q \overline{\operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C})} \xrightarrow{\sim} A^{p,q}.$$

We also have to consider the real part

$$\operatorname{Hom}_{\mathbb{R}}(T,\mathbb{R}) \subset \operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C})$$

and more general

$$\bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{R}) \subset \bigwedge^m \operatorname{Hom}_{\mathbb{C}}(T,\mathbb{C}).$$

The space T carries a real symmetric positive definit bilinear form $g = \operatorname{Re} h$. Recall the real bilinear form g on T induces an $\mathbb{R}$-isomorphimus $T \to \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ and hence by transport a bilinear form on $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$. This form induces real symmetric positive definit bilinear forms $(\cdot, \cdot)$ on

$$A^m_{\mathbb{R}} := \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R}).$$

We recall from the definition that if $U_1, \dots, U_{2n}$ is an orthonormal basis of $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ that then the

$$U_{a_{i_1}} \wedge \dots\dots U_{a_{i_m}}, \quad 1 \le i_1 < \cdots i_m \le 2n$$

perform an orthonormal basis of $A^m_{\mathbb{R}}$.

Recall that the scalar product and the natural orientation of T define a distinguished element $\omega \in \bigwedge^{2n} \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$. It is

$$\omega = U_1 \wedge \dots U_{2n}$$

in terms of an oriented orthonormal basis. Actually

$$\omega \in A^{n,n}.$$

This can be seen for example by using a basises. We choose a (complex) basis $e_1, \dots, e_n$ of T, which is orthonormal with respect to h. The dual basis in $\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})$ is denoted by $Z_1, \dots, Z_n$, hence $Z_i(e_j) = \delta_{ij}$. A basis of $\overline{\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})}$ is given by $\bar{Z}_1, \dots, \bar{Z}_n$. A real basis of $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ is $X_1, Y_1, \dots, X_n, Y_n$, where

$$X_i = \operatorname{Re} Z_i, \quad Y_i = \operatorname{Im} Z_i.$$

This basis is oriented and orthonormal with respect to g. Hence the distinguished element is

$$\omega = X_1 \wedge Y_1, \dots, X_n \wedge Y_n = \frac{1}{(-2\mathrm{i})^n} \, Z_1 \wedge \bar{Z}_1 \wedge \dots \wedge Z_n \wedge \bar{Z}_n.$$

The star operator $* : \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R}) \to \bigwedge^{2n-m} \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ is defined by

$$a \wedge b = (a \wedge *b) \, \omega.$$

We extend the star operator $\mathbb{C}$-linear to $\bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C})$. We want to extend the scalar product $(\cdot, \cdots)$ on A^m. There are two possibilities. On can extend it to a $\mathbb{C}$-bilinear form or one can extend it to a Hermitian form. We take the second choice and denote the Hermitian extension again by $(\cdot, \cdot)$. Then the star operator can be characterized by

$$a \wedge \bar{b} = (a \wedge *b) \, \omega.$$

The star operator defines actually an operator

$$* : A^{p,q} \longrightarrow A^{n-p,n-q}.$$

This can be seen for example by using basises. For this we introduce the notations

$$Z_a := Z_{a_1} \wedge \ldots \wedge Z_{a_p}$$

for a subset a of p elements $a \subset \{1, \ldots, n\}$, which are written in their natural ordering. Similarly $\bar{Z}_a$ is defined and finally

$$W_a := Z_{a_1} \wedge \bar{Z}_{a_1} \wedge \ldots \wedge Z_{a_p} \wedge \bar{Z}_{a_p}.$$

1.1 Lemma. *Let a, b, c be three disjoint subsets from $\{1, \ldots, n\}$. Then* bSb

$$*(Z_a \wedge \bar{Z}_b \wedge W_c) = \mathrm{i}^{\sharp a - \sharp b}(-1)^{m(m+1)/2 + \sharp c}(-2\mathrm{i})^{m-n}\; Z_a \wedge \bar{Z}_b \wedge W_d,$$

where $m = \sharp a + \sharp b + \sharp c$ and where d denotes the complement of $a \cup b \cup c$.

We have to consider the alternating form $\Omega = \operatorname{Im} h$. Recall that there is a canonical isomorphism

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R}) \stackrel{\sim}{\longrightarrow} \operatorname{Alt}_{\mathbb{R}}(T \times T, \mathbb{R})$$

This extends $\mathbb{C}$-linearly to

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C}) \stackrel{\sim}{\longrightarrow} \operatorname{Alt}_{\mathbb{R}}(T \times T, \mathbb{C}).$$

Hence Ω can be considered as element of $\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C})$. Actually

$$\Omega \in \bigwedge^{1,1} \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C}),$$

as for example the formula

$$\Omega = \frac{\mathrm{i}}{2} \sum_{i=1}^{n} Z_i \wedge \bar{Z}_i$$

shows. This element is fundamental in what follows. It defines an operator

$$L : A^{p,q} \longrightarrow A^{p+1,q+1}, \quad L(u) := \Omega \wedge u.$$

It is correlated with the operator

$$\Lambda = *^{-1} L * : A^{p+1,q+1} \longrightarrow A^{p,q}.$$

We also can consider L and Λ as operators

$$L : A^m \longrightarrow A^{m+1}, \quad \Lambda : A^m \longrightarrow A^{m-1}.$$

1.2 Lemma. *The operators L and Λ are adjoint,* oLL

$$(L\alpha, \beta) = (\alpha, \Lambda\beta).$$

Moreover L and Λ are real operators (i,e. they preserve $A^{\bullet}_{\mathbb{R}}$).

Proof. Check this with a real oriented orthonormal basis $U_1, \ldots U_{2n}$. □

1.3 Definition. *A form $\alpha \in A^m$ is called* ***primitive****, if $\Lambda(\alpha) = 0$.* DpF

Because of the adjointness we have

$$A^m = \mathrm{Kernel}(\Lambda) + \mathrm{Image}(L) \qquad \text{orthogonal decomposition).}$$

Hence every α can be written in the form $\alpha = \alpha_0 + L(\beta)$ with a primitive α, Repeating this argument for β we obtain

1.4 Proposition. *Every $\alpha \in A^m$ admits a decomposition* diP

$$\alpha = \sum_{2t \le m} L^t(\alpha_t)$$

with primitive $\alpha_t \in A^{m-2t}$.

Corollary. *If every primitive form in degree $\le m$ vanishes, then $A^m = 0$.*

One may ask whether the α_t are unique and how they can be computed from α. This needs some preperation. We introduce an operator C which simply has the effect

$$Cu = \mathrm{i}^{p-q}u \quad \text{for} \quad u \in A^{p,q}.$$

and another operator

$$w := C^2 = **\,.$$

The following relations are easily checked by means of a basis:

1.5 Lemma. *One has* Kre

(a) $$[L, C] = [\Lambda, C] = 0$$
(b) $$[\Lambda, L]u = (n - m)u \quad \textit{for} \quad u \in A^m.$$

.

Corollary. $(\Lambda L^k - L^k\Lambda)u = k(n - k - m + 1)L^{k-1}u$ *for* $u \in A^m$*.*

We apply this relation to a primitive form u. We notice $L^{2n+1-m}u = 0$ because we are out of the range. From the corollary we see $L^{2n-m}u = 0$ because the factor in front $(2n+1-m)n$ is different from zero. We can apply this argument again and repeat this as long the factor in front is different form 0. We come down to $u = L^0u =$ if $m > n$.

1.6 Proposition. *Primitive forms of degree $m > n$ are zero.* PDz

We will use this in the proof of the Kodaira vanishing theorem.

2. Kaehler metrics

Let (X, h) be a Hermitian manifold of pure (complex) dimension n. Recall that this is a complex analytic manifold such that the tangent bundle is equipped with a positive definit Hermitian metric. Recall that the imaginary part of h can be considered as a differential form

$$\Omega \in A^{1,1}(X), \quad \Omega = \bar{\Omega}.$$

In the special case, where X is an open subset $U \subset \mathbb{C}^n$, the Hermitean metric is given by a Hermitian matrix $h(z) = (h_{\mu\nu})$ and one has

$$\Omega = \mathrm{i}/2 \sum h_{\mu\nu}(z) dz_\mu \wedge d\bar{z}_\nu.$$

It may happen that this differential form is closed, for example when $h(z)$ is constant. It turns out that this is a very important property.

2.1 Definition. *A Kähler manifold is a Hermitian manifold, such that Ω is closed.* DKM

2.2 Proposition. *Let X be a Kähler manifold. For every point $a \in X$ there exists a holomorphic chart φ around a which maps $a \in U_\varphi$ to $0 \in V_\varphi$ and such that the corresponding Hermitian matrix $h(z)$, $z \in V_\varphi$, satisfies the following condition:* KlE

$$h_{\mu\nu}(0) = \delta_{\mu\nu}, \quad \frac{\partial h}{\partial x_i}(0) = \frac{\partial h}{\partial y_i}(0) = 0.$$

We mention by the way that this property also implies that X is Kählerian, because this property implies that $d\Omega$ is zero in the point a. But a is arbitrary. The point is that d involves only first partial derivatives.

We derive some relations between the operators $L\alpha = \Omega \wedge \alpha$ and the derivative operators $\partial, \bar{\partial}$ and the coderivative operators $\partial^*, \bar{\partial}^*$. We prefer the notation L^* instead of Λ for the adjoint operator.

2.3 Theorem. *On a Kähler manifold the following relations hold:* KRa

$$\begin{aligned} &[L, \partial] = [L, \bar{\partial}] = [L^*, \partial^*] = [L^*, \bar{\partial}^*] = 0 \\ &[L, \partial^*] = \mathrm{i}\bar{\partial}, \quad [L, \bar{\partial}^*] = -\mathrm{i}\partial \\ &[L^*, \partial] = \mathrm{i}\bar{\partial}^*, \quad [L^*, \bar{\partial}] = -\partial^*. \end{aligned}$$

Proof. These formulae can be checked directly in the case

$$\Omega = \mathrm{i}/2 \sum dz_\nu \wedge d\bar z_\nu .$$

Because they obtain only first derivatives they follow in general by 2.2. □

There can be derived some relations which also involve second derivatives, (which could not be proven directly by using 2.2). For example

$$\partial\bar\partial^* = \partial(-\mathrm{i}[L^*, \partial]) = -\mathrm{i}\partial L \partial .$$

In the same manner one proves:

2.4 Corollary. *One has* Ckr

$$\begin{aligned}\partial\bar\partial^* &= -\bar\partial^*\partial = -\mathrm{i}\bar\partial^* L \bar\partial^* = -\mathrm{i}\partial L^* \partial\\ \bar\partial\partial^* &= -\partial^*\bar\partial = \mathrm{i}\partial^* L \partial^* = \mathrm{i}\bar\partial L^* \bar\partial .\end{aligned}$$

Recall that the Laplacians on a Hermitian manifold are defined by

$$\Delta = dd^* + d^*d, \quad \Box = \partial\partial^* + \partial^*\partial, \quad \bar\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial .$$

A formal consequence of the above Kähler relations is

2.5 Theorem. *On a Kähler manifold the relations* LcL

$$\Delta = 2\Box = 2\bar\Box$$

hold.

2.6 Main theorem of Hodge theory for Kähler manifolds. *For a compact Kaehler manifold X one has in addition to III.6.1* KHt

$$H^m(X, \mathbb{C}) \cong \bigoplus_{p+q=m} H^{p,q}(X).$$

Morover

$$H^{p,q}(X) \cong H^{q,p}(X).$$

This implies for the Betti- and Hodge numbers the following relations:

$$b^m = \sum_{p+q=m} h^{p,q}, \quad h^{p,q} = h^{q,p} = h^{n-p,m-q}.$$

There are many important consequences: For example b^m is even for odd m. Moreover $b^1 = 2h^{1,0}$ hence the dimension of the space of holomorphic differentials is a topological invariant.

3. The canonical connection of a holomorphic bundle

We want to carry over part of the Kähler identities to bundle valued differential forms. Let $E \to X$ be a holomorphic bundle over a complex analytic manifold. Our problem is that we could define

$$\bar\partial : A^{p,q}(X,E) \longrightarrow A^{p,q+1}(X,E))$$

but up to know there is no operator ∂. We will see that there is a natural on if E carries a Hermitian metric.

It is worthwhile to start with some generalities about connections: We consider a differentiable vector bundle $E \to X$ over a differentiable manifold X. In principle E can be thought to be real or complex. For simplicity we restrict to complex bundles which are more important for us. Recall that we introduced bundle valued differential forms $A^m(X,E)$. In the case $m = 0$ these are the differentiable sections of E. We denote them here by

$$E^\infty(X) = A^0(X,E).$$

There is a natural map

$$A^p(X) \times E^\infty(X) \longrightarrow A^p(X,E), \quad (\omega, s) \longmapsto \omega \otimes s,$$

which is defined fibrewise and even more general

$$A^p(X) \times A^q(X,E) \longrightarrow A^{p+q}(X,E), \quad (\omega, s) \longmapsto \omega \wedge s.$$

3.1 Definition. *Let $E \to X$ be a differentiable vector bundle over a differentiable manifold. A* ***connection*** *is a family of $\mathbb{C}$-linear maps* DZu

$$D : E^\infty(X) = A^0(U,E) \longrightarrow A^1(U,E), \qquad U \subset X \ \textit{open},$$

which is compatible with restriction to smaller open subsets and such that

$$D(fs) = df \otimes s + fD(s),$$

where $f \in \mathcal{C}^\infty(U)$ is a differentiable function and $s \in E^\infty(U)$.

The connection can be extended to various types of tensors. We only need:

3.2 Lemma. *Let D be a connection on $E \to X$. There is a unique extension to a family of linear maps* LeC

$$D : A^p(U,E) \longrightarrow A^{p+1}(U,E), \qquad U \subset X \ \textit{open},$$

which is compatible with restriction to smaller open subsets and such that

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge D(s)$$

for $\omega \in A^p(U)$ and $s \in E^\infty(U)$.

Proof. It is sufficient to prove the existence and uniqueness for small open U, which are contained in the domain of definition of a chart and such that $E_U \to U$ is trivial. A little calculation shows

$$d(f\omega)\otimes s + f\omega\wedge D(s) = d\omega\otimes fs + \omega\wedge D(fs).$$

Hence the claim follows from the following little algebraic lemma:

Let R be a commutative ring with unit and $R^n \times R^m \to R^{(m,n)}$ the map $(a,b)\mapsto (a_i b_j)$ (dyadic product). Let $\beta : R^n\times R^m \to A$ be a $\mathbb{Z}$-bilinear map into some abelian group such that

$$\beta(ra,b)=\beta(a,rb) \qquad (r\in R,\ a\in R^n,\ b\in R^n).$$

Then the bilinear map factors through a homomorphism $R^{(m,n)}\to A$.

We consider

$$D^2 : E^\infty(U)\to A^2(U,E).$$

The basic fact is that this map is not zero in general but has the weaker property

$$D^2(fs)=fD^2(s)$$

for differentiable functions $f\in\mathcal{C}^\infty(U)$. This implies that D^2 is induced by a bundle map. Hence $R := D^2$ can be considered as en element of a certain Hom-bundle, namely the bundle with fibre

$$\mathrm{Hom}_{\mathbb{C}}\Big(E_a,\ \mathrm{Hom}_{\mathbb{R}}\big(\Lambda^2T_a(X),\mathbb{C}\big)\otimes_{\mathbb{C}} E_a\Big).$$

This can be reinterpreted as follows. Let A,B be two vector spaces. There are canonical isomorphisms

$$\mathrm{Hom}(A,B\otimes A)=A^*\otimes B\otimes A = B\otimes(A^*\otimes A)=B\otimes\mathrm{Hom}(A,A).$$

Hence we can consider R as a global section of the bundle

$$\mathrm{Hom}_{\mathbb{R}}\big(\Lambda^2T_a(X),\mathbb{C}\big)\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(E_a,E_a).$$

This is a bundle valued differential form.

3.3 Definition. *Let D be a connection in a differentiable vector bundle* DCk
$E\to X$. The ***curvature form*** *of D is $R=D^2$, which can be considered as bundle valued differential form*

$$R\in A^2(X,\mathrm{Hom}(E,E)).$$

A famous connection is the so called Levicivita conenction on a Riemannian manifold X. It is a certain connection in the real tangent bundle. In this case the curvature tensor is a section of the bundle

$$\operatorname{Hom}\Big(T_a(X),\ \operatorname{Hom}_{\mathbb{R}}\big(\Lambda^2 T_a(X),\mathbb{R}\big)\otimes_{\mathbb{R}} T_a(X)\Big).$$

This space can be embedded for example into $T_a(X)^{\otimes 4}$, which leads to the usual Riemann curvature tensor. For us a variant is of importance. Consider now a holomorphic vector bundle E over a complex analytic variety. Recall that in this case we have a decomposition

$$A^1(X,E) = A^{1,0}(X,E)\oplus A^{0,1}(X,E).$$

This gives us a decomposition of a connection D into two parts,

$$D = D' + D'',\quad D' : E^\infty(X)\to A^{1,0}(X,E),\ D'' : E^\infty(X)\to A^{0,1}(X,E).$$

More generally the extension 3.2 of D to $A^{p,q}(X,X)$ decomposes as well into a sum $D = D' + D''$, where

$$D' : A^{p,q}(X,E)\to A^{p+1,q}(X,E),\quad D'' : A^{p,q}(X,E)\to A^{p,q+1}(X,E).$$

We now assume that E carries a Hermitian metric $\langle\cdot,\cdot\rangle$. This induces pairings

$$\langle\cdot,\cdot\rangle : E^\infty(X)\times E^\infty(X)\longrightarrow \mathcal{C}^\infty(X).$$

More general there is a pairing

$$\langle\cdot,\cdot\rangle : E^\infty(X)\times A^1(E,X)\longrightarrow A^1(X).$$

It is induced by the obvious map

$$E_a\times E_a\otimes V\longrightarrow V,\quad \langle a, b\otimes v\rangle = \langle a,b\rangle v.$$

3.4 Proposition. *Let E be a holomorphic vector bundle over a complex analytic manifold. Assume that E is equipped with a Hermitian metric. Then there exists a unique connection —called the* ***canonical connection****— such that* CkZ

(a) $d\langle s,t\rangle = \langle Ds,t\rangle + \langle s,Dt\rangle$ *for* $s,t\in E^\infty(U)$ *(U open).*

(b) *If $s\in E^\infty(U)$ is a holomorphic section, then $D''(s) = 0$.*

Morover $D'' = \bar\partial$ on $A^{p,q}(X,E)$.

If V is a one-dimensional vector space over the field K then canonically $\mathrm{Hom}(V,V)=K$. Hence for a connection on a line bundle the curvature can be considered as a usual differential form

$$R\in A^2(X).$$

3.5 Proposition. *Let L be a holomorphic line bundle over a complex analytic manifold. Assume that L carries a Hermitian metric h. Let R be the curvature of the canonical connection. Then* KcK

$$R=\partial\bar\partial\log h,$$

hence $R/2\pi\mathrm{i}$ represents the Chern class of L (more precisely its image in $H^2(2,\mathbb{R})$).

We mention without proof that for any differentiable complex line bundle over an arbitrary differentiable manifold the class of R in $H^2(X,\mathbb{R})$ is the same for all connections.

4. Kodaira's vanishing theorem

Let $\omega\in A^{1,1}(X)\cap A^2_{\mathbb{R}}(\mathbb{R})$ be differential form on a complex analytic manifold. We recall that ω_a can be considered as a alternating bilinear form

$$\omega : T_a(X)\times T_a(X).$$

We recall that then there exists a unique Hermitian form on T_aX with imaginary part ω_a.

4.1 Definition. *A holomorphic line bundle on a complex analytic manifold is called **positive**, if there exists a real closed differential form* Hlp

$$\omega\in A^{1,1}(X)\cap A^2_{\mathbb{R}}(\mathbb{R}),$$

which represents the image of the Chern class of L and such that the corresponding Hermitian form on T_aX is positive definit for every $a\in X$.

The choice of such a form ω then induces a structure as Hermitian manifold (X,h) on X and this clearly is a Kähler structure.

4.2 Proposition. *Let L be a positive line bundle over a compact complex analytic manifold X and ω a differential form as in 4.1. There exists a Hermitian metric H on L such that* ClD

$$\omega=\mathrm{i}\partial\bar\partial\log H.$$

We choose an arbitrary Hermitian metric H of L. Hence we have a metric H on the bundle and a Kähler metric h on X. We define $\omega_0 = \mathrm{i}\partial\bar\partial \log H$. We know already that ω_0 and ω define the same class in $H^2(X, \mathbb{R})$. We want to modify H such they get equal. We can replace H by $e^h H$ with a real differentiable function h. Hence we have to find h such that $\omega_0 - \omega = \mathrm{i}\partial\bar\partial h$, This global version of III.1.4 is proved as follows.

$$\omega_0 - \omega = d\eta \quad \eta \in A^1_{\mathbb{R}}(X).$$

We decompose $\eta = \alpha + \bar\alpha$ with $\alpha \in A^{1,0}$. From $d\eta \in A^{1,1}(X)$ we obtain $\partial\alpha = 0$ and $\bar\partial\alpha + \partial\bar\alpha = d\eta$. Using Hodge theory we obtain $\alpha = \alpha_0 + \partial f$ with a $\Box$-harmonic α_0 and a function f. Now we use in an essential manner that X is Kählerien. The form α_0 is also $\bar\Box$-harmonic and hence $\bar\partial\alpha_0 = 0$. This gives $d\eta = \bar\partial\partial f + \partial\bar\partial\bar f$. Set $h = \mathrm{i}(\bar f - f)$. $\Box$

4.3 Kodaira vanishing theorem. *Let L be a positive (holomorphic) line bundle on a compact complex analytic manifold. Then* KVT

$$H^{p,q}(X, L) = 0 \quad \textit{for} \quad p + q > n.$$

Proof. We choose a Hermitian metric H on the bundle X and a Hermitian metric h on the manifold X which are tied together in the sense that $\mathrm{i}\partial\bar\partial \log H$ is the imaginary part of h (which can be considered as differential form as we explained). We want to make use of the canonical connection (3.4)

$$D = D' + D".$$

Recall that $D" = \bar\partial$ is defined in a naiv way and that the operator D' depends on the use of the Hermitian metric H and should be considered as substitute for ∂ in the vector valued case. We use for simplicity the notation $D' = \partial$. We need the adjoint operators of $\partial, \bar\partial$ with respect to the scalar product on $A^{p,q}(X, L)$

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \natural\beta,$$

where

$$\natural := \bar{*} \otimes \sharp : A^{p,q}(X, L) \longrightarrow A^{n-p,n-q}(X, L^*).$$

Recall that $\sharp : E \to E^*$ is a the antilinear isomorphism induced by the Hermitian form H on L and finally $\wedge$ is the natural pairing

$$A^{p,q}(X, E) \times A^{n-p,n-q}(X, E^*) \xrightarrow{\wedge} \mathcal{C}^\infty(X).$$

We already realized that the adjoint of $\bar\partial$ is

$$\bar\partial^* = -\sharp\bar\partial\sharp.$$

Similarly the adjoint of ∂ (“$=\partial$”) is computed as

$$\partial^* = -*\bar\partial *.$$

We also have the operator

$$L : A^{p,q}(X,L) \to A^{p+1,q+1}(X,L), \quad L(\alpha) = \Omega \wedge \alpha.$$

where Ω is the Kähler form of h. Recall that Ω is also the curvature form of D. The adoint of L is

$$L^* = -*L*.$$

For the proof of the vanishing theorem we have to make use of the identity

$$L^*\bar\partial - \bar\partial L^* = -\mathrm{i}\partial^*.$$

This identity follows from the analogue identity in the case of the trivial bundle L.

4.4 Basic identity. *Let Ω be the Kähler form of h. Then* BId

$$\partial\bar\partial + \bar\partial\partial = -\mathrm{i}L$$

or equivalently

$$\bar\partial^*\partial^* + \partial^*\bar\partial^* = -\mathrm{i}L^*.$$

Proof. This follows from the fact that Ω is the curvature form of D. Recall that the curvature form is defined by D^2. □

The proof of the vanishing theorem now is very short. Let $\omega \in A^{p,q}(X,L)$ be a $\bar\Box$-harmonic form. Then $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$. Using the above formula for ∂^* we obtain

$$\begin{aligned}\langle\partial^*\omega,\partial^*\omega\rangle &= \langle\mathrm{i}(L^*\bar\partial - \bar\partial L^*)\omega, \partial^*\omega\rangle\\ &= \langle -\mathrm{i}\bar\partial L^*\omega, \partial^*\omega\rangle = \langle -\mathrm{i}L^*\omega, \bar\partial^*\partial^*\omega\rangle\\ &= \langle -\mathrm{i}L^*\omega, (\bar\partial^*\partial^* + \partial^*\bar\partial^*)\omega\rangle.\end{aligned}$$

Using the basic identity we get

$$\langle\partial^*\omega,\partial^*\omega\rangle = -\langle \mathrm{i}L^*\omega, \mathrm{i}L^*\omega\rangle.$$

Hence both sides have to be zero, especially $L^*\omega = 0$. Hence ω is a primitive form of degree $p+q > n$. Such forms are zero (1.6). This proves the vanishing theorem. □

5. Blowing up

Let $V=\mathbb{C}^n$ be the standard complex vector space. Recall that $P(V)$ is the set of lines (=one-dimensional sub-vector spaces) in V. We consider in $V\times P(V)$ the subset

$$\pi:\hat{V}_0:=\big\{\,(v,L);\quad v\in L\,\big\}.$$

There is a natural projection $\hat{V}_0\to V$. We determine the fibres $\pi^{-1}(a)$. This fibre consists of precisely one point if $a\ne=0$, namely the point $(a,[a])$. The fibre over 0 consists of all $(0,L)$ and hence can be identified with the full $P(V)$. Hence we can have a natural bijection

$$\hat{V}_0=V-\{0\}\ \cup\ P(V).$$

The point is now that $\hat{V}_0$ is a smooth subset of the complex manifold $V\times P(V)$ and carries hence a structure as a complex analytic manifold. We are not so interested in the fact about the smoothness and define the structure directly by giving charts. We may assume $V=\mathbb{C}^n$. We consider in $P^{n-1}(\mathbb{C})=P(\mathbb{C}^n)$ the set open subset $P_i^{(n-1}(\mathbb{C})$ consisting of all $[z_1,\dots,z_n]$ such that $z_i\ne 0$. We denote by $\hat{V}_i$ its intersection with $\hat{V}_0$, which is open because of course $\hat{V}_0$ has been equipped with the induced topology. We construct a bijective map

$$\hat{V}_i\ \overset{\sim}{\longrightarrow}\ \mathbb{C}^n.$$

For simplicity of notation we assume $i=1$. Then the map in the converse direction is given by

$$(w_1,\dots,w_n)\longmapsto\big((w_1,w_1w_2,\dots,w_1w_n),[1,w_2,\dots,w_n]\big).$$

It is clear that to of these charts are holomorphic compatible. Hence we obtain a (connected) pure n-dimensional complex analytic manifold. The projection $\hat{V}_0\to V$ clearly is holomorphic.

We generalize this construction slightly. Let U be a complex analytic manifold, which is biholomorphic equivalent to an open subset of $V=\mathbb{C}^n$. Let $a\in U$ be a point. Consider the disjoint union

$$\hat{U}_a=U-\{a\}\ \cup\ P^{n-1}(\mathbb{C}).$$

We choose an open embedding $U\hookrightarrow V$ such that a is mapped to zero. There is a natural map

$$\hat{U}_a\longrightarrow\hat{V}_0,$$

which is the natural inclusion outside a and the identity on $P^{n-1}(\mathbb{C})$. It is clear that $\hat{U}_a$ is mapped bijectively to an open subset of $\hat{V}_0$. We equip $\hat{U}_0$ with the structure as complex analytic manifold such that it is mapped biholomorphically to its image. This structure does not depend on the choice of the embedding $\hat{U} \longrightarrow \hat{V}$.

We generalize this construction to an arbitray complex analytic manifold X with a distinguished point a. We define

$$\hat{X}_a := X - \{a\} \,\cup\, P^{n-1}(\mathbb{C}) \qquad (n = \dim_a X).$$

Next we consider an open neighborhood U of a which is biholomorphic equivalent to an open subset of $\mathbb{C}^n$. We have the subsets $X - \{a\}$ and $\hat{U}_a$ of $\hat{X}$. We call a subset of $\hat{X}$ open if its intersections with both subsets are open and we call a function on this subset holomorphic if its restrictions to the two intersections are holomorphic. It is clear that $\hat{X}$ gets a complex analytic manifold, that the structure does nor depend on the choice of U and that the natural map $\pi : \hat{X}_a \to X$ is holomorphic. We call $\hat{X}$ the **blow up** of X at a. The blow up construction sometimes also is called a **quadratic transformation**.

6. Maps into the projective space

Let L be a line bundle on a complex analytic manifold and $\mathcal{L}$ its sheaf of holomorphic sections. Assume that a finite system of global sections

$$s_0, \dots, s_n \in \mathcal{L}(X)$$

is given. We consider the set X_0 of all points $x \in X$ such that at least on $s_i(x)$ is different from zero. Recall that $s_i(x)$ is element of the vector space $\mathcal{L}_x$. The set X_0 is open. Let $U \subset X_0$ be an open subset over which $\mathcal{L}$ is trivial. Choosing a (holomorphic) trivialization $\mathcal{L}_U \cong U \times \mathbb{C}$, we can define $(s_0(x), \dots, s_n(x))$. Changing the trivialization means to multiply $(s_0(x), \dots, s_n(x))$ with a joint factor. Hence the point

$$[s_0(x), \dots, s_n(x)] \in P^n(\mathbb{C}).$$

is independent of the choice of the local trivialization. This means that we obtain a map

$$X_0 \longrightarrow P^n(\mathbb{C}).$$

Clearly this is a holomorphic map. The famous Kodaira vanishing theorem states that under certain circumstances $X_0 = X$ and moreover that

$$X \longrightarrow P^n(\mathbb{C})$$

is a closed embedding, i.e. a biholomorphic map onto a closed complex analytic submanifold. The proof of the vanishing theorem rests on the possibility to define cohomological obstructions for the existence of enough global sections. Here "blowing up" comes into the game. The whole principle will clear if we work out a condition that $\mathcal{L}$ admits a global section $s \in \mathcal{L}$ such that $s(a) \neq 0$ for a given point a. For this we consider the blow up

$$\hat{X} = \hat{X}_a \longrightarrow X$$

and the pull-back

$$\hat{L} \to \hat{X}$$

of the line bundle L. From its construction we get a natural linear map $\mathcal{L}(X) \longrightarrow \hat{\mathcal{L}}(\hat{X})$.

6.1 Lemma. *Let X be a complex analytic manifold, $\hat{X} \to X$ the blow up in* BuI
a point and $L \to X$ a holomorphic line bundle. The natural map

$$\mathcal{L}(X) \longrightarrow \hat{\mathcal{L}}(\hat{X})$$

is an isomorphism.

Proof. We can assume the dimension is > 1. Then we have to apply an elementary result of complex analyis, which we state without proof (absence of isolated singularities in more than one variable):

Let $U \subset \mathbb{C}^n$, $n > 0$ be an open subset and $a \in U$ a point. Every holomorphic function on $U - \{0\}$ extends to a holomorphic function on U.

A consequence of this remark is: Let $L \to X$ be a holomorphic line-bundle and s a holomorphic section over $X - \{a\}$. Then s extends to a global holomorphic section. The implies 6.1. □

Now we consider a short exact sequence (compare IV.6.3).

$$0 \longrightarrow \mathcal{J}_Y \hat{\mathcal{L}} \longrightarrow \hat{\mathcal{L}} \longrightarrow j_* \mathcal{O}_Y \longrightarrow 0$$

and the corresponding long exact sequence. A holomorphic function on a compact connected complex analytic manifold is constant. Hence

$$j_* \mathcal{O}_Y(X) = \mathcal{O}_Y(Y) = \mathbb{C}.$$

Using 6.1 we get the exact sequence

$$\mathcal{L}(X) \longrightarrow \mathbb{C} \longrightarrow H^1(X, \mathcal{J}_Y \hat{\mathcal{L}}).$$

It may happen that $H^1(X, \mathcal{J}_Y \hat{\mathcal{L}})$ vanishes. In this case we obtain that $\mathcal{L}(X) \to \mathbb{C}$ is surjective and hence the existence of a non-trivial global section. If one looks how this map is defined on sees:

6.2 Lemma. *Let X be a complex analytic manifold, $\hat{X} \to X$ the blow up in a* Egs
point a and $L \to X$ a holomorphic line bundle. We denote by Y the exceptional fibre in the blow up $\hat{X}$ of X in a. Assume $H^1(\hat{X}, \mathcal{J}_Y \hat{\mathcal{L}}) = 0$. Then there exists a global section of L which doesn't vanish in a.

7. The Kodaira embedding theorem

A connected compact complex analytic manifold X is called a **Hodge manifold** if there exist a Kähler metric such that the corresponding Kähler class is integral, i.e. in the image of $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. We know already that X is a Hodge manifold if and only if there exists a positive line-bundle on X.

7.1 Kodaira's embedding theorem. *Every Hodge manifold is biholomorphic equivalent to a complex analytic submanifold of some projective space.* KET

In this context one should mention that by Chow's theorem every complex analytic submanifold of a projective space is algebraic. Hence Hodge manifolds are "projective algebraic":

The idea is to use the techniques of the previous section and to embed a Hodge manifold by means of a positive line-bundle L into some projective space. It is easy to work out the conditions that L leads to an embedding. We need a notation. Let $f : U \to \mathbb{C}$ a holomorphic function on an open subset $U \subset \mathbb{C}^n$. We say that f *vanishes at least in second order* in a point a, if $f(a) = 0$ and if all partial derivatives vanish. It is clear that this definition extends to complex analytic manifolds and moreover makes sense for sections of holomorphic line-bundles. This can also be expressed in an invariant manner. Let L be a line bundle and $\mathcal{L}$ its sheaf of holomorphic sections. A section $s \in \mathcal{L}(U)$ vanishes in a point $a \in \mathcal{L}(U)$ in at least in second order, if its stalk s_a is contained in $\mathfrak{m}_a^2\mathcal{L}_a$, where $\mathfrak{m}_a \subset \mathcal{O}_a$ denotes the ideal of germs which vanish in a.

7.2 Definition. *A holomorphic line bundle L on a compact complex analytic manifold is called **strict ample**, if the following three conditions are stisfied.* DsA

(C1) *For every point $a \in X$ there exist a global (holomorphic) section s, which doesn't vanish in a.*
(C2) *Point seperation: If a, b are two different points, then there exists a global section s with $s(a) = 0$, $s(b) \neq 0$.*
(C3) *Infinitesimal seperation: Let $t \in \mathcal{L}(U)$ be a holomorphic section in some neighborhood of a point a. There exists a global section s such that $s - t$ vanishes in at least second order in a.*

*The line bundle is called **ample**, if there is some tensor power $L^{\otimes m}$, which is strict ample.*

The condition (C1) can of course be cancelled. We left it, because it is the basis starting condition. It implies that

$$X \longrightarrow P^m(\mathbb{C}), \quad x \longmapsto [s_0(x), \dots, s_m(x)]$$

is a everywhere defined holomorphic map, where $s_0, \dots, s_m$ denotes a basis of the space of global sections. The condition (C2) says that this map is injective and (C3) shows that the tangent map is injective everywhere. Because X is compact, the above map is a topological map from X onto its image. Now it is clear that the image os smooth an that X is mapped biholomorphically onto its image. Kodaira's embedding theorem hence follows from

7.3 Theorem. *Positive line bundles are ample.* Pla

In the previous section we formulated already a sufficient cohomological condition for (C1), namely $H^1(\hat{X}, \mathcal{J}_Y\hat{\mathcal{L}}) = 0$ for all blow ups $\hat{X}$ of X in arbitrary point, Y denotes the exceptional fibre (=inverse image of a in $\hat{X}$). It is easy to derive similar conditions for (C2) and (C3). We skip details and formulate what comes out. We change a little bit the notations: Recall that the set of isomorphy classes of holomorphic line bundles is an abelian group $\mathrm{Pic}(X)$. The composition is induced by the tensor product. We didn't construct the product in the language of $\mathcal{O}_X$-modules because it would involve tensor products over rings. Let $Y \subset X$ be a smooth submanifold of a complex analytic manifold. We simply denote the image of the ideal sheaf $\mathcal{J}_Y$ by $[Y]$ and we denote also by $[L]$ the image of a line bundle in $\mathrm{Pic}(X)$. Finally we use the sign "+" for the composition in $\mathrm{Pic}(X)$. The image of $\mathcal{J}_Y\hat{\mathcal{L}}$ in $\mathrm{Pic}(\hat{X})$ is

$$[Y] + [\hat{L}] \qquad \text{(replaces } \mathcal{J}_Y\hat{\mathcal{L}}).$$

7.4 Lemma. *A line bundle L is strict ample if the following conditions are satisfied:* saC

- (C1') *Let $\hat{X} \to X$ the blow up of X in a point. Denote by Y the exceptional fibre and by $\hat{L}$ the pull back of L. The first cohomology of $[Y] + \hat{L}$ vanishes.*
- (C2') *Let a_1, a_2 be two different points of X and $\hat{X} \to X$ the blow up of X in the two points a, b. Let Y_1, Y_2 be the two exceptional fibres and $\hat{L}$ the pull back of L. The first cohomology of $[Y_1] + [Y_2] + \hat{L}$ vanishes.*
- (C3') *Let $\hat{X} \to X$ the blow up of X in a point. Denote by Y the exceptional fibre and by $\hat{L}$ the pull back of L. The first cohomology of $[Y] + [Y] + \hat{L}$ vanishes.*

Now the Kodaira vanishing theorem comes into the game. We denote by $K_X \in \mathrm{Pic}(X)$ the image of the canonical bundle of a compact complex analytic manifold X. By the vanishing theorem the first cohomology of a line bundle L vanishes, if $[L] - K_X$ is positive.

7.5 Lemma. *A line bundle L is ample if there exists a natural number m such that the following conditions are satisfied: In the notations of 7.4 the following (classes of) bundles* saP

$$[Y] + m[\hat{L}] - K_{\hat{X}}, \quad [Y_1] + [Y_2] + m[\hat{L}] - K_{\hat{X}}, \quad 2[Y] + m[\hat{L}] - K_{\hat{X}}$$

are positive:

Before we can apply this lemma we need some information about $K_{\hat{X}}$.

7.6 Lemma. *Let $Y \subset \hat{X}$ be the exceptional fibre of the blow-up $\hat{X} \to X$ of a n-dimensional complex analytic manifold in one or more points. Denote by $\hat{K}_X$ the pull-back of the canonical bundle. Then* Kbb

$$K_{\hat{X}} = \hat{K}_X - (n-1)[Y].$$

Now we are ready for

The proof of the embedding theorem

We will construct a natural number m_0 such that for all $m \geq m_0$ the bundles

$$m[\hat{L}] + n[Y] - \hat{K}_X, \quad m[\hat{L}] + n[Y_1] + n[Y_2] - \hat{K}_X, \quad m[\hat{L}] + 2n[Y] - \hat{K}_X$$

are positive for arbitrary choices of the points which are blown up. The point is that m_0 has to be independent of these points. The proofs in all three cases are very similar. We restrict to the first one. We have to construct a bundle metric on (a realization of the class) $m[\hat{L}] + n[Y] - \hat{K}_X$. We need the following principle, whose simple proof is omitted:

Let (L_1, h_1), (L_2, h_2) be two line bundles with Hermitian metrics. Then $L_1 \otimes L_2$ carries an obvious tensor product metric $h = h_1 \otimes h_2$. One has

$$\partial\bar{\partial} \log(h_1 \otimes h_2) = \partial\bar{\partial} \log h_1 + \partial\bar{\partial} \log h_2.$$

The bundle metric on $m[\hat{L}] + n[Y] - \hat{K}_X$ is constructed as tensor product of metrics on $\hat{L}$, $\hat{K}_X$, $[Y]$ separately. The bundle L already carries a positive metric. We choose an arbitrary metric on K_X. In this way we get a metric on $m[L] + K_X$. We use now the trivial fact that for sufficiently large $mH + H'$ will be a positive definit form if H' is an arbitrary and H a positive definit form on some vector space. Hence the Hermitian form on T_aX of the metric on $m[L] - K_X$ is positive definit if m is big enough, $m \geq m_0$. The bound for m nay be depend on a. A simple continuity and compactness argument shows that m_0 can be chosen independent of a. Now we take the pull back metric on $m[\hat{L}] - K_{\hat{X}}$, $m \geq m_0$. The corresponding Hermitian form on the tangent spaces $T_a\hat{X}$ will be positive definit outside the exceptional fibre. Inside the exceptional fibre we only can say that it is semi-definit.

We will remedy this situation using $[Y]$. The metric on $[Y]$ has to be constructed carefully (because Y varies). What we need is a metric on $[Y]$ with the following properties. The associated Hermitian form on the tangent spaces is positive semi-definit every where. We first consider an suitable chosen open neighborhood $Y \subset W \subset \hat{X}$ of the exceptional fibre and construct a metric of $[Y]|W$. We recall the blow-up construction. We can W identify with a submanifold of $B \times P^{n-1}(\mathbb{C})$, where B is the open ball $\|z\| < 1$ in $\mathbb{C}^n$, which should be considered as an open neighborhood of a in X.

$$W \hookrightarrow B \times P^{n-1}(\mathbb{C}).$$

On $[W]$ we can consider the (restriction of the line-bundle) $[L]$. There is another line bundle on W. Consider the basic line bundle on $P^{n-1}(\mathbb{C})$ which belongs to a linear subspace $P^{n-2}(\mathbb{C})$. This line bundle can be pulled back to a line bundle on $B \times P^{n-1}(\mathbb{C})$ by means of the natural projection. Then it can be restricted to W. A simple local computation shows:

7.7 Lemma. *Let $Y \subset W \subset \hat{X}$ be an open neighborhood, which is biholomorphic equivalent to a submanifold of $B \times P^{n-1}(\mathbb{C})$ (blow-up construction). Here $a \in B$*subsetX *is an open neighborhood which is biholomorphic equivalent to a ball. The line bundle $[Y]|W$ corresponds to the basic bundle on $P^{n-1}(\mathbb{C})$ in the following sense: $[Y]$ on W is isomorphic to the line bundle, which is obtained by restriction to W the pull-back of the basic bundle on $P^{n-1}(\mathbb{C}$ to $B \times P^{n-1}(\mathbb{C})$.* bsP

The basic bundle on $P^{n-1}(\mathbb{C})$ carries a metric whose associated Hermitian form is positive definit. We pull back this bundle an its metric to $B \times P^{n-1}(\mathbb{C})$. We consider the associated Hermitian forms on the tangent spaces $T_bB \times T_aP^{n-1}(\mathbb{C})$. This Hermitian form is the pull back of a positive definit Hermitian form on T_bB.

We also can consider the pull-back $\tilde{L}$ of our line bundle L to $B \times P^{n-1}(\mathbb{C})$ by means of the natural projection to B. The metric of L pulls back to a metric of $\tilde{L}$. Its Hermitian form is the pull-back of a positive definit Hermitian form on T_bB. The two pulled back Hermitian forms are only positive semi-definit. But there sum is clearly definit! Now we restrict to W and obtain.

The bundle $[Y] + n\hat{L}$ is positive on the neighborhood W of the exceptional fibre.

Now we consider a metric of $[Y]$ on the whole $\hat{X}$ whose restriction to a may be smaller neighborhood of the exceptional fibre agrees with the metric constructed above. It is not difficult to construct such a metric using partition of unity arguments. The bundle $[Y] + n\hat{L}$ needs not to be positive on the whole $\hat{X}$. But since $\hat{L}$ is positive outside the exceptional fibre we obtain that $m[Y] + n\hat{L}$ is positive of large $m \geq 0$, It follows that $m[\hat{L}] + n[Y] - \hat{K}_X$ is positive for $m \geq 0$.

We still have to verify that the bound m_0 is independent of the base point of the blow-up. This follows from a continuity and compactness argument for $\hat{X} \times \hat{X}$ (one component representing the base point).

The completes the embedding theorem. □