THE GEOMETRY AND ARITHMETIC OF A CALABI-YAU SIEGEL THREEFOLD

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1. Introduction

In the recent papers [3] and [4], the two last named authors gave two constructions of Siegel modular varieties that admit a Calabi-Yau model. They used two different methods, but the crucial step in both constructions is to consider an action of a finite group G, fixing a holomorphic three form, on a projective variety M. In the first case the variety M was Igusa's (toroidal) compactification of the Siegel modular variety of level 4, in the second case they started from a Siegel modular variety $\mathcal{X}$ introduced by van Geemen and Nygaard, cf. [5].

The main advantage of the first method is that it gives really projective Calabi-Yau manifolds. The second one produces a bigger class of manifolds, however these examples need not to be projective. They are "Calabi-Yau manifolds" in the sense that they are (smooth) compact complex manifolds with trivial canonical bundle and vanishing first Betti number. Combining the two approaches we get an intermediate class of projective examples corresponding to all groups between $\Gamma_2[2] \cap \Gamma_{2,0}[4]$ and $\Gamma_{2,0}[2]_{\mathbf{n}}$ (Theorem 2).

A careful analysis of the first method leads to a modular variety $\mathcal{Y}$ of particular interest related to a significant modular group, namely the group

$$\Gamma = \Gamma_2[2] \cap \Gamma_{2,0}[4].$$

Since Γ is the minimal group allowed in the above theorem all constructed Calabi–Yau manifolds can be realized as smooth models of quotients of $\mathcal{Y}$ by a finite group.

The aim of this paper is to treat in details the modular variety $\mathcal{Y}$ and its Calabi-Yau model. As in most constructions of Calabi–Yau threefolds the two main difficulties are to prove the existence of a projective crepant resolution and to compute the Hodge numbers (Theorem 10).

Date: February 10, 2011.

*Partially supported by MNiSW grant no. N N201 388834.

We shall describe the geometry of $\tilde{\mathcal{Y}}$ using two approaches. The first one is based on the construction of $\mathcal{Y}$ as a Siegel variety and was developed in [4], where the action of G on the Picard group of the regular locus of $\mathcal{X}$ has been determined. This method allows to compute the Euler number and Picard number of many examples constructed using Theorem 2. We mention that both are bimeromorphic invariants. Hence these numbers can be computed using an arbitrary model, even a non-projective one. We will come back to this in an extra paper.

The second method is purely algebraic geometric. Using the equations (1) derived from the ring of modular forms we can describe $\mathcal{Y}$ as a complete intersection of type $(4,4)$ in the weighted projective space $\mathbb{P}(1,1,1,1,2,2)$ or as a iterated double covering of the projective space $\mathbb{P}^3$. Now, the results of [2] allows us to give an explicit description of a projective Calabi–Yau model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ and compute its Euler number and the dimension of the space of infinitesimal deformations. Contrary to the first method where we descend the Hodge numbers from a variety to its quotient here we lift invariants from a variety to its covering.

In the first method we compute directly the Picard group whereas in the second one we study the space of infinitesimal deformations. Consequently in the first approach we go down from van Geemen's and Nygaard's variety to its quotient and in the second we go up from projective space $\mathbb{P}^3$ which is a quotient of $\mathcal{Y}$.

Representation of $\mathcal{Y}$ as a complete intersection in a weighted projective space exhibits a K3 fibration on the Calabi–Yau manifold $\tilde{\mathcal{Y}}$. Exceptional divisors of the blow–ups in the resolution of singularities together with components of irreducible fibers of the K3 fibration give generator set of the Picard group.

The Calabi–Yau manifold $\tilde{\mathcal{Y}}$ has also very interesting arithmetic properties, as a rigid Calabi–Yau threefold defined over $\mathbb{Q}$ it is modular, we proved that the corresponding modular form is the unique newform of weight 4 and level 8. We show that $\mathcal{Y}$ is birational to a $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ quotient of the self–fiber product of the Beauville surface associated to the group $\Gamma_{1,0}[4]\cap\Gamma_1[2]$.

All examples constructed with Theorem 2 are birational to a quotient of $\mathcal{Y}$ and so they contain a rank 2 subrepresentation in the middle étale cohomology isomorphic to the representation associated to the above cusp form. Since we have an algebraic correspondence explaining this isomorphism they are all "relatives" in the sense of [9].

2. Modular varieties

As in [4] the starting point of our investigation is the variety

$$\mathcal{X}: \qquad \begin{array}{l} Y_0^2 = X_0^2 + X_1^2 + X_2^2 + X_3^2 \\ Y_1^2 = X_0^2 - X_1^2 + X_2^2 - X_3^2 \\ Y_2^2 = X_0^2 + X_1^2 - X_2^2 - X_3^2 \\ Y_3^2 = X_0^2 - X_1^2 - X_2^2 + X_3^2 \end{array}$$

This is a modular variety, in the sense that it is biholomorphic to the Satake compactification of $\mathbb{H}_2/\Gamma'$ for a certain subgroup $\Gamma' \subset \mathrm{Sp}(4, \mathbb{Z})$. For details, we refer to [5], [1] and [4], we just recall the basic informations that we need.

Let $\mathbb{H}_n$ be the Siegel upper half space of symmetric complex matrices with positive definite imaginary part. The symplectic group $\Gamma_n := \mathrm{Sp}(2n, \mathbb{Z})$ acts on $\mathbb{H}_n$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}.$$

Here we think of elements of Γ_n as consisting of four $n \times n$ blocks. For any subgroup of finite index $\Gamma \subset \Gamma_n$ the Satake compactification $\overline{\mathbb{H}_n/\Gamma}$ of the quotient $\mathbb{H}_n/\Gamma$ is the projective variety associated to a graded algebra of modular forms. We recall briefly its definition. A modular form f of weight $r/2$, $r \in \mathbb{N}$, is a holomorphic function f on $\mathbb{H}_n$ with the transformation property

$$f(MZ) = v(M)\sqrt{\det(CZ + D)}^{\,r} f(Z)$$

for all $M \in \Gamma$. In the case $n = 1$ a regularity condition at the cusps has to be added. Here $v(M)$ is a multiplier system. Essentially it fulfills the cocycle condition. We denote this space by $[\Gamma, r/2, v]$. Fixing some starting weight $r_0/2$ and a multiplier system v for it, we define the ring

$$A(\Gamma) := \bigoplus_{r \in \mathbb{N}} [\Gamma, rr_0/2, v^r].$$

This turns out to be a finitely generated graded algebra and its associated projective variety $\mathrm{Proj}(A(\Gamma))$ can be identified with the Satake compactification. The ring depends on the starting weight and the multiplier system but the associated projective variety does not.

The simplest examples of modular forms are given by theta constants. A characteristic is an element $m = \binom{a}{b}$ from $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Here $a, b \in (\mathbb{Z}/2\mathbb{Z})^n$ are column vectors. The characteristic is called even if ${}^t ab = 0$ and odd otherwise. The group $\mathrm{Sp}(2n, \mathbb{Z}/2\mathbb{Z})$ acts on the set of characteristics by

$$M\{m\} := {}^t\!M^{-1}m + \begin{pmatrix} (A\,{}^t\!B)_0 \\ (C\,{}^t\!D)_0 \end{pmatrix}.$$

Here S_0 denotes the column built of the diagonal of a square matrix S. It is well-known that $\mathrm{Sp}(2n,\mathbb{Z}/2\mathbb{Z})$ acts transitively on the subsets of even and odd characteristics. Recall that to any characteristic the theta function

$$\vartheta[m] = \sum_{g\in\mathbb{Z}^n} e^{i\pi(Z[g+a/2]+{}^tb(g+a/2))} \qquad (Z[g] = {}^t\!gZg)$$

can be defined. Here we use the identification of $\mathbb{Z}/2\mathbb{Z}$ with the subset $\{0,1\}\subset\mathbb{Z}$. It vanishes if and only if m is odd. Recall also that the formula

$$\vartheta[M\{m\}](MZ) = v(M,m)\sqrt{\det(CZ+D)}\vartheta[m](Z)$$

holds for $M\in\Gamma_n$, where $v(M,m)$ is a rather delicate eighth root of unity which depends on the choice of the square root. Sometimes, when $n=2$, we will use the notation

$$\vartheta[m] = \vartheta\begin{bmatrix} a_1a_2 \\ b_1b_2 \end{bmatrix} \quad \text{for} \quad m = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}.$$

We consider the 8 functions

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix}(Z),\quad \vartheta\begin{bmatrix}00\\10\end{bmatrix}(Z),\quad \vartheta\begin{bmatrix}00\\01\end{bmatrix}(Z),\quad \vartheta\begin{bmatrix}00\\11\end{bmatrix}(Z),$$

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix}(2Z),\quad \vartheta\begin{bmatrix}10\\00\end{bmatrix}(2Z),\quad \vartheta\begin{bmatrix}01\\00\end{bmatrix}(2Z),\quad \vartheta\begin{bmatrix}11\\00\end{bmatrix}(2Z).$$

If we denote them by $Y_0,\dots,Y_3, X_0\dots,X_3$, then classical addition formulas for theta constants show that the relations defining $\mathcal{X}$ hold. These eight forms are modular forms of weight $1/2$ for a group Γ' that we are going to define.

We set

$$\begin{aligned}
\Gamma_n[q] &= \operatorname{kernel}(\Gamma_n \to \mathrm{Sp}(2n,\mathbb{Z}/q\mathbb{Z})),\\
\Gamma_n[q,2q] &= \{M\in\Gamma_n[q];\quad (A\,{}^t\!B/q)_0 \equiv (C\,{}^t\!D/q)_0 \equiv 0 \bmod 2\},\\
\Gamma_{n,0}[q] &= \{M\in\Gamma_n;\ C\equiv 0 \bmod q\},\\
\Gamma_{n,0,\vartheta}[q] &= \{M\in\Gamma_{n,0}[q]\quad (C\,{}^t\!D/q)_0 \equiv 0 \bmod 2\}.
\end{aligned}$$

Here S_0 denotes the diagonal of the matrix S.

The group Γ', which belongs to van Geemen's and Nygaard's variety is defined by

$$\Gamma' = \{M \in \Gamma_2[2,4] \cap \Gamma_{2,0,\vartheta}[4]; \quad \det D \equiv \pm 1 \mod 8\}.$$

We are going to recall the main result of [4]. The group $\Gamma_{n,0}[q]$ can be extended by the Fricke involution

$$J_q = \begin{pmatrix} 0 & E/\sqrt{q} \\ -\sqrt{q}E & 0 \end{pmatrix}.$$

We denote by $\hat{\Gamma}_{2,0}[2]$ the extension of $\Gamma_{2,0}[2]$ by J_2, i.e.

$$\hat{\Gamma}_{2,0}[2] = \Gamma_{2,0}[2] \cup J_2\Gamma_{2,0}[2].$$

$\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$ is a subgroup of index two of $\hat{\Gamma}_{2,0}[2]$ that is the kernel of a character $\chi_{\mathbf{n}}$. The character χ_n is given on $\Gamma_{2,0}[2]$ as the product of the unique nontrivial character for the full modular group and the character

$$M \mapsto (-1)^{(\alpha+\beta+\gamma)/2}, \text{ for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{0,2}[2], \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = C^t D$$

and extended to $\hat{\Gamma}_{2,0}[2]$ by leting $\chi_n(J_2) = 1$ (see [3]). We also have to consider the group $\Gamma_{2,0}[2]_{\mathbf{n}} = \hat{\Gamma}_{2,0}[2]_{\mathbf{n}} \cap \Gamma_{2,0}[2]$. With these notations we have from [4]:

Theorem 1. *The Satake compactification of the Siegel modular threefold which belongs to a group between* Γ' *and* $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$*, admits a desingularization in the category of complex spaces that admits a holomorphic three form without zeros and whose first Betti-number is zero.*

There are thousands of conjugacy classes of intermediate groups. As has been pointed out by van Geemen not all desingularizations can be constructed as projective varieties. So we get not always Calabi-Yau varieties in the usual sense. (By a Calabi-Yau threefold we understand a three-dimensional smooth complex *projective* variety which admits a holomorphic differential form of degree three without zeros and such that the first Betti number is zero.) Using an extension of the methods of [3] we obtain a smaller class of groups but with projective models. We describe them in the next section.

3. The variety $\mathcal{Y}$

There is one intermediate group of particular interest, namely the group

$$\Gamma = \Gamma_2[2] \cap \Gamma_{2,0}[4].$$

This group contains Γ' as subgroup of index 32. It is stable under the Fricke involution J_2. For this group (and as a consequence for all groups between it and $\Gamma_{2,0}[2]_{\mathbf{n}}$) we can use the methods of the paper [3] to construct a (projective) Calabi-Yau model.

Let $\tilde{X}(4)$ be the Igusa desingularization of the Satake compactification of $\mathbb{H}_2/\Gamma_2[4]$, for a description we refer to the original paper [7]. It has several holomorphic three forms. In [3] it has been described the zero locus of a suitable one, $\omega_{\mathbf{n}}$. This results to be the ramification locus of the quotient of $\tilde{X}(4)$ for a group that we are going to describe. In the regular part of the quotient the form has no zeros, so we have

Theorem 2. *Let* $\tilde{X}(4)$ *be the Igusa desingularization of the Satake compactification of* $\mathbb{H}_2/\Gamma_2[4]$*. Then the quotient* $\tilde{X}(4)/(\Gamma_2[2]\cap\Gamma_{2,0}[4])$ *admits a desingularization which is a Calabi-Yau manifold. The same is true for all groups between* $\Gamma_2[2]\cap\Gamma_{2,0}[4]$ *and* $\Gamma_{2,0}[2]_{\mathbf{n}}$*.*

For a proof we proceed as it follows. A careful, but simple analysis, leads us to consider translation matrices

$$T_S = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$$

of level two, $S \equiv 0 \bmod 2$. Such a translation matrix is called *reflective* if S is congruent mod 4 to one of the three

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Actually reflective translations act as reflections on the Igusa desingularization of level four.

Lemma 3. *The group* Γ *is generated by*

1) *The group* $\Gamma_2[4]$*,*

2) *The elements of* $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$*, which are conjugate inside* Γ_2 *to the diagonal matrix with diagonal* $(1,-1,1,-1)$*.*

3) *All elements of* $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$*, which are conjugate inside* Γ_2 *to a reflective translation matrix* $\left(\begin{smallmatrix} E & S \\ 0 & E \end{smallmatrix}\right)$ *of* $\Gamma_2[2]$*.*

The proof can be easily done with the help of a computer.

The lemma is similar to lemma 1.4 in [3]. There the group $\Gamma_{2,0}[2]_{\mathbf{n}}\cap\Gamma_2[2]$ has been characterized by the same properties 1)–3) with the only difference that the word "reflective" has been skipped. The same proof as in [3] works with this weaker assumption and gives the result that the quotient of the Igusa desingularization for the principal congruence subgroup of level four $\tilde{X}(4)/(\Gamma_2[2]\cap\Gamma_{2,0}[4])$ admits a desingularization that is a Calabi-Yau manifold. The same then is true for any group between Γ and $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$.

By standard method (going down process) we can produce the structure of the ring of modular forms of this distinguished case:

Proposition 4. *The ring of modular forms of even weight for the group* $\Gamma_{2.0}[4] \cap \Gamma_2[2]$ *is generated by*

$$\theta\begin{bmatrix}00\\00\end{bmatrix}\theta\begin{bmatrix}00\\01\end{bmatrix}\theta\begin{bmatrix}00\\10\end{bmatrix}\theta\begin{bmatrix}00\\11\end{bmatrix},$$

all pairs of the form

$$\theta\begin{bmatrix}00\\ab\end{bmatrix}^2\theta\begin{bmatrix}00\\cd\end{bmatrix}^2$$

and the ten even $\theta[m]^4$.
If one wants the generators also in the odd weights, it is enough to add the form of weight 3

$$T=\theta\begin{bmatrix}10\\00\end{bmatrix}\theta\begin{bmatrix}10\\01\end{bmatrix}\theta\begin{bmatrix}01\\00\end{bmatrix}\theta\begin{bmatrix}01\\10\end{bmatrix}\theta\begin{bmatrix}11\\00\end{bmatrix}\theta\begin{bmatrix}11\\11\end{bmatrix}.$$

To simplify the equations we consider the ring of forms of even weights:

Proposition 5. *The ring* $A(\Gamma_{2,0}[4] \cap \Gamma_2[2])^{(2)}$ *in the even weights is equal to*

$$\mathbb{C}\left[\theta\begin{bmatrix}00\\00\end{bmatrix}\theta\begin{bmatrix}00\\01\end{bmatrix}\theta\begin{bmatrix}00\\10\end{bmatrix}\theta\begin{bmatrix}00\\11\end{bmatrix},\theta\begin{bmatrix}00\\00\end{bmatrix}^2,\theta\begin{bmatrix}00\\01\end{bmatrix}^2,\theta\begin{bmatrix}00\\10\end{bmatrix}^2,\theta\begin{bmatrix}00\\11\end{bmatrix}^2,y_4\right]^{(2)}$$

with

$$y_4=-\theta\begin{bmatrix}10\\01\end{bmatrix}^4-\theta\begin{bmatrix}00\\11\end{bmatrix}^4$$

Denoting the above variables by y_5, x_0, x_1, x_2, x_3 we have the ring

$$\mathbb{C}[y_5, x_0, x_1, x_2, x_3, y_4]^{(2)}$$

with x_i of weight 1 and y_j of weight 2. We have also the following defining relations

$$(1)\qquad \begin{aligned} y_5^2 &= x_0x_1x_2x_3,\\ 2y_5^2 &= x_0^2x_1^2+x_0^2x_3^2+x_1^2x_3^2+(-x_2^2+x_0^2+x_1^2+x_3^2+y_4)y_4. \end{aligned}$$

We shall denote by $\mathcal{Y}$ the modular variety defined by the above equations.

We want to explain how we can compute the Hodge numbers of a Calabi–Yau model of the variety $\mathcal{Y}$ without the description of an explicit crepant resolution.

For this we need some information about the group $K := \Gamma/\Gamma'$. The basic information is that K is abelian of order 32 and that all elements are of order two. So their fixed point loci are known from [4]. From this paper we also know that K extends to a suitable small resolution $\tilde{\mathcal{X}}$, where this resolution is a complex manifold and not necessarily projective. The quotient $\tilde{\mathcal{X}}/K$ admits a crepant resolution. This is not a Calabi-Yau model in the usual sense, since it is not apriori projective, but it is bimeromorphic equivalent to any true (projective) Calabi-Yau model. Now a theorem of Kollar (theorem 4.9 in [8]) states that the the models —even the non-projective ones— are related by flops. As a consequence they have the same Betti numbers (4.12 in [Ko]) and also the same Picard number. (The Picard number is defined to be the rank of the group of divisor classes.) Hence we can compute the Euler number and the Picard number from the may not projective model.

We also know that the fixed point locus is a curve $C \subset \tilde{\mathcal{X}}$. The image of C in $\tilde{\mathcal{X}}/K$ is the singular locus. The local structure of a singularity is of the type $\mathbb{C}^3/A$, where A either is a group of order 2, generated by a transformation, which changes two signs or the group of order 4 which contains all sign changes at two positions. It is easy to describe the crepant resolution for these singularities (see [3]) and from this description on can see:

Lemma 6. *The number of exceptional divisors of a crepant resolution of $\tilde{\mathcal{X}}/K$ equals the number of irreducible components of the fixed point locus of K on $\mathcal{X}$, modulo K.*

One can check that K contains 6 elements which have nodes as isolated fixed points. Each of them fixes 16 nodes. So each node occurs as fixed point of K. Hence all 96 exceptional lines on $\tilde{X}$ are in the fixed point locus of K. There are exactly 12 orbits under the action of the group K. Now we have to count only the one dimensional fixed curves in $\mathcal{X}$. This can be done with the results of [4]. We just give the result: There are 12 elements of K having a one dimensional fixed point locus and each of them has 4 components, which are elliptic curves, and form the two K-orbits.

Lemma 7. *The number of components of the fixed point locus of K on $\tilde{\mathcal{X}}/K$ is* 36.

Now we are able to compute the Picard number of a Calabi-Yau model of $\mathcal{X}/K$. The Picard number of the regular locus can be computed by means of the results of section 6, especially theorem 6.4 in [4]. The result of a computation is 4. Hence we get:

Lemma 8. *The Picard number of a Calabi-Yau model of $\mathcal{Y}$ is* 40.

Let us compute the Euler number. We recall that the crepant resolution $\tilde{\mathcal{X}}$ has Euler number equal to 64. Since K is abelian, the string theoretic formula ([10, Thm. 2]) gives

$$e(\tilde{\mathcal{Y}}) = \frac{1}{32} \sum_{(g,h)\in K\times K} e(\tilde{\mathcal{X}}^{<g,h>}) =$$

$$\frac{64}{32} + \frac{3}{32}\sum_{g\neq id} e(\tilde{\mathcal{X}}^{g}) + \frac{1}{32}\sum_{\mathrm{id}\neq g\neq h\neq id} e(\tilde{\mathcal{X}}^{<g,h>})$$

Since the fixed point set of a single involution is an elliptic curve or one of the 96 exceptional lines, we get

$$e = 20 + \frac{1}{32}\sum_{\mathrm{id}\neq g\neq h\neq id} e(\tilde{\mathcal{X}}^{<g,h>}).$$

We still have to discuss how for two different g, h, which are different from the identity, the fixed point loci intersect in $\tilde{\mathcal{X}}$. We want to compare this with the intersection of the fixed point loci on the singular model $\mathcal{X}$. We have to discuss two cases,

- g fixes a curve in $\mathcal{X}$ and h fixes a node.
- *Both g and h have one dimensional fixed point locus on $\mathcal{X}$ (4 elliptic curves).*

In the first case there are 12 g which fix a curve and 6 h which fix a node. Hence we have 72 cases to consider. In 48 cases the intersection of the fixed point loci in $\mathcal{X}$ is empty. Hence only 24 pairs are of interest. In each case the fixed locus $\mathrm{Fix}(g)$ of g is the union of 4 smooth elliptic curves

$$\mathrm{Fix}(g) = E_1 \cup E_2 \cup E_3 \cup E_4.$$

and the fixed point locus of of h consists of 16 nodes. The intersection of $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ consists of 8 nodes. Each single E_i contains 4 of these 8 nodes. This shows that in each of the 8 nodes two of the 4 elliptic curves come together. Now we consider $\tilde{\mathcal{X}}$. Since the fixed point set of g is smooth, it consists of four elliptic curves $\tilde{E}_1, \dots, \tilde{E}_4$, such that the natural projection $\tilde{E}_i \to E_i$ is biholomorphic. Let a be one of the 8 nodes in $\mathrm{Fix}(g) \cap \mathrm{Fix}(h)$. We can assume that E_1, E_2 are the two elliptic curves which run into a. Let C be the exceptional line over a. Then g induces an automorphism of C of order two. Since an involution P^1 has two fixed points, we see that $\tilde{E}_1$ and $\tilde{E}_2$ each hit C in one intersection point and both points are different. So each of the 8 exceptional lines carries two intersection points. This shows:

Lemma 9. *Let $g \in K$ be an element with a one dimensional fixed point set, and $h \in K$ an element, which fixes nodes. There are* 24 *possibilities. The joint fixed point locus on $\tilde{\mathcal{X}}$ consists of* 16 *points.*

In the formula for the Euler number each pair (g,h) of the above form contributes with 16/32. We have 24 pairs. Together with the pairs (h,g) we get the contribution 24 to the Euler number. Hence we have

$$e = 44 + \frac{1}{32} \sum_{\substack{\mathrm{id}\neq g\neq h\neq \mathrm{id} \\ dim\mathrm{Fix}(g)=dim\mathrm{Fix}(h)=1}} e(\tilde{\mathcal{X}}^{<g,h>}).$$

In the second case, both g and h have one dimensional fixed point locus on $\mathcal{X}$ (4 elliptic curves). The number of intersection points of $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ on $\mathcal{X}$ is 0, 8 or 16. The number of pairs (g,h) with 8 intersection points is 24 and that with 16 intersection points is 48.

Let us consider pairs with 16 intersection points. In this case one can check that none of the 16 is a node, and one can check furthermore that the contribution to the Euler for each such pair is $(1/32)\cdot 16 = 1/2$.

Now we consider pairs with 8 intersection points. In this case one can check that all 8 intersection points are nodes. Let a be such a node. One can see that that two of the components of $\mathrm{Fix}(g)$ run into a and the same is true for $\mathrm{Fix}(h)$. Moreover a simple computation gives that gh has a as isolated singularity. Hence as in the first case above g has two fixed points a_1, a_2 on the exceptional line C over a and h has the same fixed points. Hence 12 is the contribution to the Euler number. We get as contribution $36 = 24 + 12$ to the Euler number. This gives

$$e = 80$$

for the Euler number.

Consequently, we have proved the following theorem

Theorem 10. *The variety $\mathcal{Y}$ has a (projective) Calabi–Yau model with Hodge numbers $h^{11} = 40$, $h^{12} = 0$.*

Remark 11. *If a Calabi–Yau manifold constructed using Theorem 2 is not rigid we can interpret geometrically part of the space of deformations with fixed curves of the quotient. A fixed curve in the resolution process produces a ruled surface E in the Calabi–Yau threefold over some curve C of genus g. If $g > 0$ then E deforms with the Calabi–Yau manifold on a submanifold of codimension g of its Kuranishi space of $\tilde{X}$, over a general point of the Kuranishi space E is replaced by a sum of $2g-2$ rational curves (see* [12]*).*

4. Explicit Calabi–Yau model of $\tilde{\mathcal{Y}}$

In this section we shall give alternative description of the Calabi–Yau manifold $\tilde{\mathcal{Y}}$ using only the equations (1) of $\mathcal{Y}$ as a complete intersection in the weighted projective space $\mathbb{P}(1,1,1,1,2,2)$. These equations allow us to consider $\mathcal{Y}$ as a $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ covering of the projective space $\mathbb{P}^3$ branched along a pair of quartic surfaces. As a consequence we are able to use the standard methods of double coverings to describe a crepant resolution of $\mathcal{Y}$, compute its Euler characteristic and Hodge numbers (via the dimension of the deformations space). We also give an explicit correspondences with the van Geemen's and Nygaard's variety and the self fiber product of Beauville's surface.

Subtracting twice the first equation in (1) from the second one and changing the coordinate system

$$(x_0,x_1,x_2,x_3,y_4,y_5)\mapsto(x_0,x_1,x_2,x_3,\tfrac{1}{2}(y_4+x_2^2-x_0^2-x_1^2-x_3^2),y_5)$$

we get the following representation of $\mathcal{Y}$ as a complete intersection in $\mathbb{P}(1,1,1,1,2,2)$

$$\begin{aligned} y_5^2 &= x_0x_1x_2x_3\\ y_4^2 &= (x_0+x_1+x_2+x_3)\times(x_0-x_1-x_2+x_3)\times\\ &\quad\times(x_0-x_1+x_2-x_3)\times(x_0+x_1-x_2-x_3) \end{aligned}$$

Description of the rings of modular forms for varieties $\mathcal{X}$ and $\mathcal{Y}$ yields the following quotient map

$$(X_0,X_1,X_2,X_3,Y_0,Y_1,Y_2,Y_3)\mapsto(Y_0^2,Y_1^2,Y_2^2,Y_3^2,16X_0X_1X_2X_3,Y_0Y_1Y_2Y_3)$$

so the action on $\mathcal{X}$ is diagonal given by the following group

$$K:=\{\varepsilon\in(\mathbb{Z}/2\mathbb{Z})^8:\varepsilon_0=1,\varepsilon_1\varepsilon_2\varepsilon_3=1,\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7=1\}\cong(\mathbb{Z}/2\mathbb{Z})^5.$$

We are going to describe an explicit crepant resolution of $\mathcal{Y}$. Variety $\mathcal{Y}$ may be considered as $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ covering of $\mathbb{P}^3$, or as an iterated double covering. The branch locus consists of two quartics

$$\begin{aligned} D_1 &= \{x_0x_1x_2x_3=0\},\\ D_2 &= \{(x_0+x_1+x_2+x_3)\times(x_0-x_1-x_2+x_3)\times\\ &\quad\times(x_0-x_1+x_2-x_3)\times(x_0+x_1-x_2-x_3)=0\}. \end{aligned}$$

Both quartics D_1 and D_2 are sums of four faces of tetrahedra in $\mathbb{P}^3$, so each of them gives four triple point and six double lines which we denote $l_1^{(1)},\dots,l_6^{(1)}$ and $l_1^{(2)},\dots,l_6^{(2)}$.

Each of the lines $l_i^{(1)}$ intersect two of the lines $l_j^{(2)}$ giving rise to 12 fourfold points of the octic $D:=D_1+D_2$, which we denote $P_1,\dots,P_{12}$.

The intersection $D_1\cap D_2$ is a sum of sixteen lines (intersections of pair of planes a component of D_1 and a component of D_2)

$$D_1\cap D_2=\sum_{i=1}^{16}C_i.$$

Let $\sigma_1 : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ be the blow–up of $\mathbb{P}^3$ at points $P_1, \dots, P_{12}$, let $\tilde{l}_j^{(i)}$ denotes the strict transform of $l_j^{(i)}$ and $\tilde{D}_i$ the strict transform of D_i. Then the lines $\tilde{l}_i^{(1)}$ and $\tilde{l}_j^{(2)}$ are disjoint whereas four triples out of $\tilde{l}_i^{(1)}$ and four triples out of $\tilde{l}_i^{(2)}$ intersect at a triple point. Moreover we have $\tilde{D}_i = \sigma_1^* D_i - 2\,\mathrm{exc}(\sigma_1)$, $K_{\widetilde{\mathbb{P}^3}} = \sigma_1^* K_{\mathbb{P}^3} + 2\,\mathrm{exc}(\sigma_1)$ hence

$$K_{\widetilde{\mathbb{P}^3}} + \tfrac{1}{2}(\tilde{D}_1 + \tilde{D}_2) = \sigma_1^*(K_{\mathbb{P}^3} + \tfrac{1}{2}(D_1 + D_2)).$$

Let $\sigma_2 : \mathbb{P}^* :\longrightarrow \widetilde{\mathbb{P}^3}$ be the composition of blow–ups of (strict transforms of) lines $\tilde{l}_j^{(i)}$. For each blow–up the strict transform of the quartic which contain it equals the pullback minus twice the exceptional divisor, whereas for the other quartic the strict transform equals the pullback.

Denote by

$$\sigma : \mathbb{P}^* \longrightarrow \mathbb{P}^3$$

composition $\sigma := \sigma_2 \circ \sigma_1$ and by D_i^* the strict transform of D_i. Then D_1^* and D_2^* are smooth divisors intersecting transversally along a disjoint sum of 16 lines.

Let $\pi : \tilde{\mathcal{Y}} \longrightarrow \mathbb{P}^1$ be a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ Galois covering of $\mathbb{P}^*$ branched along divisors D_1^* and D_2^*, denote $D^* = D_1^* + D_2^*$.

Lemma 12.

$$\begin{aligned}
\pi_* \mathcal{O}_{\tilde{\mathcal{Y}}} &= \mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}(D_1^* + D_2^*)) \oplus \mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*) \oplus \mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}D_1^*) \oplus \mathcal{O}_{\mathbb{P}^*}. \\
\pi_* \Theta_{\tilde{\mathcal{Y}}} &= \Theta_{\mathbb{P}^*}(-\tfrac{1}{2}D^*) \oplus \Theta_{\mathbb{P}^*}(\log D_1^*)(-\tfrac{1}{2}D_2^*) \oplus \\
&\quad \oplus \Theta_{\mathbb{P}^*}(\log D_2^*)(-\tfrac{1}{2}D_1^*) \oplus \Theta_{\mathbb{P}^*}(\log D^*). \\
K_{\tilde{\mathcal{Y}}} &= 0.
\end{aligned}$$

Proof. The first two assertion can be directly verified in local coordinates, they also follows from factoring the map π into a composition of two double covering: double covering of $\mathbb{P}^*$ branched along D_1^* followed by a double covering branched along pullback of D_2^* (or similar with D_1 and D_2 exchanged). From this factorization it follows that $K_{\tilde{\mathcal{Y}}} = K_{\mathbb{P}^*} + \frac{1}{2}(D_1^* + D_2^*) = \pi^*(K_{\mathbb{P}^3} + \frac{1}{2}(D_1 + D_2)) = 0$.

□

Now, we can give another proof of Theorem 10. Since the map σ is a composition of blow–ups with smooth centers $\sigma_* \mathcal{O}_{\mathbb{P}^*} = \mathcal{O}_{\mathbb{P}^3}$ and $R^i \sigma_* \mathcal{O}_{\mathbb{P}^*} = 0$, for $i > 0$. So by the Leray spectral sequence and Serre duality $H^1(\mathcal{O}_{\mathbb{P}^*}) = H^1(\mathcal{O}_{\mathbb{P}^3}) = 0$ and $H^1(\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}(D_1^* + D_2^*))) = H^1(K_{\mathbb{P}^*}) = H^2(\mathcal{O}_{\mathbb{P}^*}) = H^2(\mathcal{O}_{\mathbb{P}^3}) = 0$.

Claim. $\sigma_*\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}D_2^*) = \mathcal{O}_{\mathbb{P}^3}(-\frac{1}{2}D_2)$, $R^i\sigma_*\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}D_2^*) = 0$ for $i > 0$. To prove the claim we shall consider every blow–up separately. Let $\mathcal{L}$ be a line bundle on a smooth threefold P and let $\tau : \tilde{P} \longrightarrow P$ be a blow–up of a smooth subvariety $C \subset P$ with an exceptional divisor E. Let $\mathcal{M}$ be a line bundle on $\tilde{P}$ satisfying one of the following three conditions

- C is a curve and $\mathcal{M} = \tau^*\mathcal{L}$,
- C is a curve and $\mathcal{M} = \tau^*\mathcal{L} \otimes \mathcal{O}_{\tilde{P}}(E)$,
- C is a point and $\mathcal{M} = \tau^*\mathcal{L} \otimes \mathcal{O}_{\tilde{P}}(E)$.

In the first case, by the projection formula, $\tau_*\mathcal{M} = \mathcal{L}$ and $R^i\tau_*\mathcal{M} = 0$. In the other two cases consider the following exact sequence

$$0 \longrightarrow \tau^*\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \tau^*\mathcal{L} \otimes \mathcal{O}_E(-1) \longrightarrow 0.$$

Since $\tau_*(\mathcal{O}_E(-1)) = R^i\tau_*(\mathcal{O}_E(-1)) = 0$, applying the direct image functor to the above exact sequence yields $\tau_*\mathcal{M} = \mathcal{L}$ and $R^i\tau_*\mathcal{M} = 0$ and the claim follows.

From the Leray spectral sequence we get

$$H^1(\mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*)) = H^1(\sigma_*\mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*)) = H^1(\mathcal{O}_{\mathbb{P}^3}(-\tfrac{1}{2}D_2)) = 0$$

and (by symmetry) $H^1(\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}D_2^*)) = 0$.

The map π is finite so using Lemma 12 we get

$$H^1(\mathcal{O}_{\tilde{\mathcal{Y}}}) = 0$$

which proves that $\tilde{\mathcal{Y}}$ is a Calabi–Yau threefold.

By the above description $\mathbb{P}^*$ is the projective space $\mathbb{P}^3$ blown–up at twelve points and twelve lines so

$$e(\mathbb{P}^*) = 4 + 12 \times 2 + 12 \times 2 = 52.$$

Observe that blowing–up a double line containing a triple point blows–up also one of the planes containing this point, whereas blowing–up a fourfold point blows–up all four planes through this point. Consequently D_1^* is a sum of four planes blown–up 28 times, so

$$e(D_1^*) = e(D_2^*) = 4 \times 3 + 28 = 40$$

and $D_1^* \cap D_2^*$ is a disjoint sum of 16 lines so

$$e(D_1^* \cap D_2^*) = 32.$$

Now,

$$e(\tilde{\mathcal{Y}}) = 4e(\mathbb{P}^*) - 2e(D_1^*) - 2e(D_2^*) + e(D_1^* \cap D_2^*) = 4\times 52 - 2\times 80 + 32 = 80.$$

To prove that $h^{1,2}(\tilde{\mathcal{Y}}) = 0$, we shall proceed as in [2]. By [2, Thm. 4.7] $H^1(\Theta_{\mathbb{P}^*}(\log D^*))$ is isomorphic to the space of equisingular deformations of D in $\mathbb{P}^3$, moreover it is isomorphic to $(I_{eq}(D)/J_F)_8$, where J_F is the jacobian ideal of D and

$$I_{eq} = \bigcap_{i=1}^{12}(I(P_i)^4 + J_F) \cap \bigcap_{i=1}^{6}\bigcap_{j=1}^{2}(I(l_j^{(i)})^2 + J_F)$$

is the equisingular ideal. Using this formula we check with Singular ([6]) that $H^1(\Theta_{\mathbb{P}^*}(\log D^*)) = 0$.

As in the resolution of $\mathcal{Y}$ we blow–up only rational curves, by [2, Prop. 5.1] $H^1(\Theta_{\mathbb{P}^*}(-\frac{1}{2}D^*)) = 0$.

Consider the following exact sequence

$$0 \longrightarrow \Theta_{\mathbb{P}^*}(\log D_1^*)(-\tfrac{1}{2}D_2^*) \longrightarrow \Theta_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*) \longrightarrow \mathcal{N}_{D_1^*}(-\tfrac{1}{2}D_2^*) \longrightarrow 0.$$

We shall study first $\Theta_{\mathbb{P}^*}(-\frac{1}{2}D_2^*)$ and again consider separately a single blow–up $\tau : \tilde{P} \longrightarrow P$ with a smooth center C and an exceptional divisor E. We have the same three cases

- C is a curve and $\mathcal{M} = \tau^*\mathcal{L}$,
- C is a curve and $\mathcal{M} = \tau^*\mathcal{L} \otimes \mathcal{O}_{\tilde{P}}(E)$,
- C is a point and $\mathcal{M} = \tau^*\mathcal{L} \otimes \mathcal{O}_{\tilde{P}}(E)$,

where τ is as before, and consider the vector bundle $\Theta_{\tilde{P}} \otimes \mathcal{M}$. Using [2, Sect. 5] in the first and third cases ($k > 0$ in notations of [2]) we get $\tau_*(\Theta_{\tilde{P}} \otimes \mathcal{M}) = \Theta_P \otimes \mathcal{L}$ and $R^i\tau_*(\Theta_{\tilde{P}} \otimes \mathcal{M}) = 0$. Since in this case $\mathcal{N}_C \otimes \mathcal{L} = K_C$, we get

$$H^1(\Theta_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*)) = 0.$$

Finally, to find $H^0(\mathcal{N}_{D_1^*}(-\frac{1}{2}D_2^*))$ consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^*}(-\tfrac{1}{2}D_2^*) \longrightarrow \mathcal{O}_{\mathbb{P}^*}(D_1^* - \tfrac{1}{2}D_2^*) \longrightarrow \mathcal{N}_{D_1^*}(-\tfrac{1}{2}D_2^*) \longrightarrow 0.$$

Since $\sigma_*(\mathcal{O}_{\mathbb{P}^*}(D_1^* - \frac{1}{2}D_2^*)) = \mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{I}$, where $\mathcal{I}$ is the ideal of functions vanishing at $P_1, \dots, P_{12}$ and vanishing to order two along $l_1^{(1)}, \dots, l_6^{(1)}$, we get $H^0(\mathcal{O}_{\mathbb{P}^*}(D_1^* - \frac{1}{2}D_2^*)) = 0$. Since $H^1(\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}D_2^*)) = 0$, we get $H^0(\mathcal{N}_{D_1^*}(-\frac{1}{2}D_2^*)) = 0$ and consequently $H^1(\Theta_{\mathbb{P}^*}(-\frac{1}{2}D^*)) = 0$. By the above exact sequence we get $H^1(\Theta_{\mathbb{P}^*}(\log D_1^*)(-\frac{1}{2}D_2^*)) = 0$ and (by symmetry) $H^1(\Theta_{\mathbb{P}^*}(\log D_2^*)(-\frac{1}{2}D_1^*)) = 0$.

Since the map π is finite Lemma 12 yields

$$\begin{aligned} H^1(\Theta_{\tilde{\mathcal{Y}}}) = {} & H^1(\Theta_{\mathbb{P}^*}(-\tfrac{1}{2}D^*)) + H^1(\Theta_{\mathbb{P}^*}(\log D_1)(-\tfrac{1}{2}D_2^*)) + \\ & + H^1(\Theta_{\mathbb{P}^*}(\log D_2^*)(-\tfrac{1}{2}D_1^*)) + H^1(\Theta_{\mathbb{P}^*}(\log D^*)) = 0 \end{aligned}$$

and by the Serre duality

$$h^{1,2}(\tilde{\mathcal{Y}}) = 0.$$

Since the Hodge numbers of a Calabi–Yau manifold $\tilde{\mathcal{Y}}$ satisfy $e(\tilde{\mathcal{Y}}) = 2(h^{1,1}(\tilde{\mathcal{Y}}) - h^{1,2}(\tilde{\mathcal{Y}}))$ we conclude

$$h^{1,1}(\tilde{\mathcal{Y}}) = 40.$$

Remark 13. *In the proof of Theorem 10 given in section 3 we computed the Picard group of $\mathcal{Y}$ using a covering $\mathcal{X} \longrightarrow \mathcal{Y}$. Here we study the space of infinitesimal deformations of $\mathcal{Y}$, so it is more natural to work in opposite direction, we we represent $\mathcal{Y}$ as a covering of $\mathbb{P}^3$.*

There is another intersection of four quadrics related to the Calabi–Yau manifold $\tilde{\mathcal{Y}}$. After the coordinate change

$$(x_0 : x_1 : x_2 : x_3 : y_4 : y_5) \mapsto (x_0 + x_1, x_0 - x_1, x_2 + x_3, x_2 - x_3 : y_4 : \tfrac{1}{2} y_5)$$

the equations are transformed into more symmetric

$$\begin{aligned} y_5^2 &= (x_0^2 - x_1^2)(x_2^2 - x_3^2), \\ y_4^2 &= (x_0^2 - x_2^2)(x_1^2 - x_3^2). \end{aligned}$$

Consequently it is a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$–quotient of the following intersection of four quadrics

$$\begin{aligned} u_0^2 &= x_0^2 - x_1^2, \\ u_1^2 &= x_1^2 - x_2^2, \\ u_2^2 &= x_2^2 - x_3^2, \\ u_3^2 &= x_3^2 - x_0^2 \end{aligned} \tag{2}$$

in $\mathbb{P}^7$. The intersection S of two quadric in $\mathbb{P}^4$

$$\begin{aligned} u_0^2 &= x_0^2 - x_1^2, \\ u_1^2 &= x_1^2 - x_2^2 \end{aligned}$$

is singular at points $(0 : 0 : 1 : 0 : \pm i)$, $(1 : 0 : 0 : \pm 1 : 0)$, the rational map $\pi : S \ni (x_0 : x_1 : x_2 : u_0 : u_1) \longrightarrow (x_0 : x_2)$ is undetermined at points $(0 : 1 : 0 : \pm i : \pm 1)$ (intersection of the surface S with the plane $x_0 = x_2 = 0$). Blowing–up S at singular points and then at points of indeterminacy yields a rational elliptic surfaces $\tilde{\pi} : \tilde{S} \longrightarrow \mathbb{P}^1$ with four singular fibers: of type I_4 at $0, \infty$ and I_2 at ± 1. It means that $\tilde{S}$ is the Beauville modular surfaces associated to the group $\Gamma_{1,0}[4] \cap \Gamma_1[2]$ and the intersection (2) is the self fiber product of $\tilde{S}$.

Remark 14. *The Calabi–Yau manifold $\tilde{\mathcal{Y}}$ is modular with the unique cusp form of weight 4 and level 8, the same as in the case of van*

Geemen's and Nygaard's complete intersection $\mathcal{X}$ ([5]) *and the self–fiber product of the Beauville surface associated to* $\Gamma_{1,0}[4] \cap \Gamma_1[2]$ *([11]). The rational map* $\mathcal{X} \longrightarrow \mathcal{Y}$ *is defined over* $\mathbb{Q}$ *and so it induces an isomorphism between the Galois representations, hence modularity of* $\tilde{\mathcal{Y}}$ *follows from the modularity of van Geemen's Nygaard's complete intersection. One can also prove that using the Faltings–Serre–Livné method. Using a computer program we verify that for p prime,* $p \leq 97$ *the number of points in* $\mathcal{Y}(\mathbb{F}_p)$ *equals*

$$1 + p^3 - a_p + 16(p + p^2) - 12(2p + p^2),$$

where a_p *is the coefficient of the cusp form.*

For any Calabi–Yau threefold constructed with Theorem 2 we have an algebraic correspondence with van Geemen's and Nygaard's variety, consequently the l–adic Galois representation associated to (a twist of) the above newform occurs in the middle cohomology (it is relative of van Geemen's and Nygaard's variety in the sense of [9]). In [1] 15 relatives of van Geemen's and Nygaard's variety are discussed and several correspondences between them are constructed.

5. K3 fibration and the Picard group

The Hodge number $h^{1,1}(\tilde{\mathcal{Y}}) = 40$ equals the Picard number of the Calabi–Yau manifold $\tilde{\mathcal{Y}}$. The resolution of singularities of $\mathcal{Y}$ yields 37 apparent linearly independent divisors:

- pullback of a hyperplane section in $\mathbb{P}^3$,
- 12 blow–ups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ covers of a plane the exceptional loci of blow–ups of fourfold points,
- 24 blow–ups of double covers of exceptional divisors of blow–up of a double line, since after blowing–up fourfold points any double line is disjoint from one of the branch divisors, the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ covers splits into a pair of double covers.

Remark 15. *The twelve fourfold points points have the form* $(1 : \pm 1 : 0 : 0 : 0 : 0)$ *and their permutations of* $x_1, \ldots, x_4$ *coordinates. By the description of the quotient map they correspond to the 12 orbits of the nodes under* K *action.*

The twelve lines $l_j^{(i)}$ *correspond by the quotient map to the intersections of* $\mathcal{X}$ *with linear subspaces* $X_k = X_l = 0$ *or* $Y_k = Y_l = 0$ *which are sums of four elliptic curves.*

So the above description of 36 linearly independent divisors agrees with the description given in Lemma 7.

In this way we can identify rank 37 subgroup in the Picard group. To identify the remaining divisors we can use one of the K3 fibrations on $\tilde{\mathcal{Y}}$. Fix a double line of one of the quartics (f.i. fix the line $m := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\} \subset D_1$) and let $P_{(s:t)} := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 : sx_2 + tx_3 = 0\}$ ($(s : t) \in \mathbb{P}^1$) be the pencil of planes that defines a fibration on $\tilde{\mathcal{Y}}$. For $(s : t) \neq 0, \infty, \pm 1$, the fiber $S_{(s:t)}$ is a smooth K3 surface, it can be described as resolution of the complete intersection in $\mathbb{P}(1, 1, 1, 2, 1)$

$$\begin{aligned} y_5^2 =& x_0x_1, \\ y_4^2 =& (tx_0 + tx_1 + (t - s)x_2) \times (tx_0 - tx_1 + (-t - s)x_2)\times \\ &\times(tx_0 - tx_1 + (t + s)x_2) \times (tx_0 + tx_1 + (-t + s)x_2). \end{aligned}$$

This is a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ covering of $\mathbb{P}^2$ branched along a pair and a quadruple of lines in general position, the branch curves have seven nodes. Described resolution of singularities of $\mathcal{Y}$ induces also a resolution of singularities of a generic fiber by blowing–up the double points of the branch curves, each of them induces two independent divisors in the Picard group. Together with a hyperplane section and one of two irreducible components of the strict transform of the line $x_2 = 0$ we get 16 linearly independent divisors.

There are however three more independent divisors, the lines $tx_0 + sx_1 = 0$ and $sx_0 + tx_1 = 0$ and the conic $tx_0x_1 + sx_2^2 = 0$ in $P_{(s:t)}$ intersects the branch divisors only with multiplicity two, so they split in the covering into four components. Taking one components from each of them shows that the Picard number of the generic fiber is at least 19, which is the biggest possible.

The singular fibers are reducible, comparing with the resolution we get:

the fiber $S_{1:1}$ (resp. $S_{1:-1}$) has three 3 components: the strict transform of the plane, divisor corresponding to the blow–up of the point $(0 : 0 : 1 : -1)$ (resp. $(0 : 0 : 1 : 1)$) and the line $x_0+x_1 = x_2+x_3 = 0$ (resp. $x_0 + x_1 = x_2 + x_3 = 0$),

the fiber $S_{(1:0)}$ (resp. $S_{(0:1)}$) has 9 components: the strict transform of the plane, four divisor corresponding to the blow–up of points $(0 : 1 : 0 : 1)$, $(0 : 1 : 0 : -1)$, $(1 : 0 : 0 : 1)$, $(1 : 0 : 0 : -1)$ (resp. $(0 : 1 : 1 : 0)$, $(0 : 1 : -1 : 0)$, $(1 : 0 : 1 : 0)$, $(1 : 0 : -1 : 0)$) four divisors (two pairs) corresponding to the lines $x_0 = x_2 = 0$ and $x_1 = x_2 = 0$ (resp. $x_0 = x_2 = 0$ and $x_1 = x_2 = 0$).

On the Calabi–Yau model the three divisors on the generic fiber of fibration corresponds to components of the strict transforms of the quadrics

$$x_0x_1 = x_2x_3,\ x_0x_2 = x_1x_3,\ x_0x_3 = x_1x_2$$

in $\mathbb{P}^3$.

Remark 16. *Since*

$$Y_0^2Y_1^2 - Y_2^2Y_3^2 = 4(X_0X_2 - X_1X_3)^2,$$

components of the strict transform of the quadric $x_0x_1 = x_2x_3$ correspond via the quotient map to the components of the intersection of $\mathcal{X}$ with the quadric $X_0X_2 - X_1X_3$. These Weil divisors on $\mathcal{X}$ are not $\mathbb{Q}$–Cartier, they give a projective small resolution of van Geemen's and Nygaard's variety (cf. [4]*).*

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