In der Vorlesung (SS2018) wird eine Einführung in die Theorie der *holomorphen* Modulformen mehrerer Veränderlicher gegeben. Sie ist eingebettet in einen allgemeinen Rahmen, welcher auf der Cartanschen Klassifikation der beschränkten symmetrischen Gebiete beruht. Diese Klassifikation wird dargestellt, in der Sprache halbeinfacher Liescher Gruppen. Beweise werden in diesem allgmeinen Teil wenig gegeben. Insbesondere wird die Theorie der Liealgebren nicht entwickelt. Wer sich für die Theorie im Einzelnen interessiert, wird auf das fundamentale Buch von Helgason "Differential geometry and symmetric spaces" verwiesen. Der Klassifikationssatz liefert (neben zwei Ausnahmegebieten, die wir nicht behandeln) die 4 klassischen Serien, welche zu den Gruppen

$$\mathrm{SU}(p,q),\quad \mathrm{SO}^*(2n),\quad \mathrm{Sp}(2n,\mathbb{R}),\ \mathrm{O}(2,n)$$

gehören. Die Vorlesung verfolgt zwei Ziele. Es soll ein allgemeiner Rahmen für die Theorie der holomorphen Modulformen gelegt werden und es sollen einige Beispiele genauer studiert werden. Die Theorie der Siegelschen Modulformen wird eingehender behandelt werden und hierin die Satakekompaktifizierung. Diese führt zu der Erkenntnis, dass der Siegelsche Modulraum eine algebraische Varietät ist. Wir müssen also auch über komplexe Räume und algbraische Varietäten reden. Hier werde ich eine Kurzeinführung ohne Beweise geben. Ein anderes, sehr interessantes Beispiel, in dem aktuell ebenfalls stark geforscht wird, sind die orthogonalen Modulformen. Die Vorlesung richtet sich an fortgeschrittene Studenten. Ich erwarte Grundkenntnisse aus verschiedenen Gebieten, wie sie in geometrischen Theorien verwendet werden. Natürlich muss nich alles parat sein. Mathematik zu treiben bedeutet "Mut zur Lücke" und ständige Bereitschaft, Lücken zu schließen.

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Modular forms of several variables

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Contents

Chapter I. Lattices in Lie groups

1. Generalities about Lie-groups

A topological group G is a group which carries also a topology such that the maps

$$G \times G \longrightarrow G,\ (g,h) \longmapsto gh, \quad G \longrightarrow G,\ g \longmapsto g^{-1},$$

are continuous. Here $G \times G$ has been equipped with the product topology. A *locally compact group* is a topological group whose underlying space is locally compact. We always assume that G has a countable basis of the topology.

Notion of a Radon-measure. Let X be a locally compact Hausdorff space with countable basis of the topology. A Radon measure dx is a linear functional

$$I : \mathcal{C}_c(X) \longrightarrow \mathbb{C}, \quad f \longmapsto I(f) = \int_X f(x)dx$$

with the properties $I(\bar{f}) = \overline{I(f)}$ and $I(f) \geq 0$ for real $f \geq 0$. Here $\mathcal{C}_c(X)$ denotes the space of complex valued continuous functions with compact support. Let $U \subset X$ be an open subset and χ_U its characteristic function. Then one can define

$$\operatorname{vol}(U) := \sup\{I(f),\ f \in \mathcal{C}_c(X),\ f \leq \chi_U\}.$$

This can be a finite number but also ∞. In particular, $\operatorname{vol}(X) \leq \infty$ is well defined.

A Haar measure on a locally compact group G is a non-zero left invariant Radon measure

$$\int_G f(gx)dx = \int_G f(x)dx \qquad (g \in G).$$

We make use of the fact that a non zero Haar measure always exists and is uniquely determined up to a constant factor. A left invariant measure needs not to be right invariant. There exists continuous homomorphism Δ of G into the (multiplicative) group of positive real numbers such that

$$\int_G f(xg)dx = \Delta(g) \int_G f(x)dx \qquad (g \in G).$$

A locally compact group is called unimodular if every left invariant Radon measure is also right invariant. This means that the function Δ is constant one.

1) Abelian groups are unimodular.
2) A group G is unimodular if its commutator subgroup is dense.
3) Compact groups are unimodular.
4) Discrete groups are unimodular.

Let $H \subset G$ be a closed subgroup of a locally compact group. Then

$$G/H := \{gH; \quad g \in G\}$$

can be equipped with the quotient topology. The natural map $G \to G/H$ is continous and G/H is locally compact and with countable basis of the topology too. The group G acts on G/H through multiplication from the left. One can ask whether there exists a Radon measure on G/H that is invariant under this action. This is not always the case. The necessary and sufficient condition is $\Delta_G|H = \Delta_H$. In this case the invariant measure is also unique up to a constant factor. The defining formula for the quotient measure is

$$\int_G f(x)dx = \int_{G/H} \int_H f(xh)dhdx, \qquad f \in \mathcal{C}_c(G).$$

To interpret this formula, one has to observe that $\int_H f(xh)dh$ is right invariant and be considered as function on G/H. It has compact support. We denoted the Haar measure on G and the quotient measure by the same letter dx. There can arise no confusion, since the subscripts of the integral signs show the integration space.

The same can be done for $H\backslash G$. A right invariant measure exists if and only if $\Delta_G|H = \Delta_H$. In the defining formula right instead of left invariant measures have to be used.

We use the notion of a Lie group. A Lie group is a locally compact group G that is equipped with a structure of a differentiable manifold (in the $\mathcal{C}^\infty$-sense) such that multiplication $G \times G \to G$ and inversion $G \to G$ are differentiable maps. Here we have to use that the cartesian product of two differentiable manifold carries a natural structure of a differentiable manifold as well. The groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ are Lie groups. There are basis results which we use without proof.

1) *Every closed subgroup of a Lie group is smooth and hence also a Lie group.*
2) *Let $H \subset G$ be a closed subgroup of a Lie group. Then the space of cosets $H\backslash G = \{Hg;\ g \in G\}$ carries a natural structure as differentiable manifold. In the case that H is normal, it is a Lie group.*
3) *A continuous homomorphism between Lie groups is differentiable.*

It follows that a topological group admits at most one structure as Lie group. For a Lie group G, we denote by G^0 the connected component that contains the unit element. This is an open and closed subgroup and a Lie group as well.

Simple and semisimple Lie Lie groups

1.1 Definition. *A connected Lie group group G is called* ***simple*** *if it has positive dimension, if it has finite center and if besides $\{e\}$ and G there is no connected closed normal subgroup.*

Simple Lie groups are not abelian. For example, the group $\mathrm{SL}(n, \mathbb{R})$, $n \geq 2$, is simple. (It is not simple as abstract group, since it contains the normal subgroup $\{\pm E\}$.)

1.2 Definition. *A not necessarily connected Lie group is called simple if it has only finitely many connected components and if its connected component G^0 is simple.*

1.3 Definition. *A connected Lie G group is called* ***semisimple*** *if it has finite center and if there exist connected simple Lie groups G_i, $i = 1, \dots n$, $n \geq 1$, and a surjective continuous homomorphism $G_1 \times \dots \times G_n \to G$ with a finite kernel.*

1.4 Definition. *A not necessarily connected Lie G group is called* ***semisimple*** *if it has only finitely many connected components and if the connected component G^0 is semisimple.*

We list some properties of semisimple Lie groups whose proof rests on the theory of Lie algebras.

1.5 Proposition. *Let G be a semisimple Lie group, then its connected component G^0 is semisimple as well. In particular, the center Z^0 of G^0 is finite. The quotient G^0/Z^0 is also a semisimple Lie group with trivial center.*

We call G^0/Z^0 the adjoint form of G. It is connected and has trivial center.

1.6 Definition. *Two semisimple Lie groups G, H are called* ***isogenous*** *if their adjoint forms are isomorphic as Lie-groups.*

1.7 Proposition. 1) Let H be an open subgroup of finite index of the Lie group G. If one of them is simple (semisimple) than also the other and both are isogenous.

2) Let Δ be a finite normal subgroup of a Lie group G. If one of the G, G/Δ is simple (semisimple), then both are and both are isogenous.

1.8 Proposition. *Let $G_1, \dots, G_n$ and $H_1, \dots, H_n$ be simple Lie groups such that $G_1 \times \dots \times G_n$ and $H_1 \times \dots H_m$ are isogenous. Then $n = m$ and after reordering the group G_i is isogenous to H_i.*

1.9 Proposition. *Let G be a connected semisimple Lie group. Then each finite normal subgroup of G is contained in the center.*

1.10 Proposition. *Semisimple Lie groups are unimodular*

The proof rests on the Iwasawa decomposition.

2. Basic examples of semisimple Lie groups

The group $\mathrm{SL}(n,\mathbb{R})$, $n \geq 2$, is simple. The group $\mathrm{GL}(n,\mathbb{R})$ is not semisimple, since there there is a big center. Also $\mathrm{SL}(n,\mathbb{C})$ is a simple Lie group. Here we consider $\mathrm{SL}(n,\mathbb{C})$ as Lie group as defined above. The underlying space is a differentiable manifold in the usual (real) sense. Later we will consider $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$ also as complex Lie groups. This means that the underlying spaces are complex analytic manifolds.

Orthogonal groups over the field of reals.

Let V be a finite dimensional real vector space and let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form. It is called non-degenerated if every $a \in V$ with the property $B(a,x) = 0$ for all $x \in V$ is zero. This is equivalent to the property that $B(x,a) = 0$ for all $x \in V$ implies that $a = 0$. The Sylvester theorem states the following.

Assume that $(\cdot,\cdot)$ is a non-degenerated symmetric bilinear group. Then there exists a basis $e_1,\dots,e_n$ such that the Gram matrix (e_i,e_j) equals

$$E_{q,q} = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

Here E_n denotes the $n \times n$-unit matrix. The signature (p,q) is uniquely determined.

The orthogonal group $\mathrm{O}(V,(\cdot,\cdot))$ is the subgroup of $\mathrm{GL}(V)$ that preserves the bilinear form. It is isomorphic to the group

$$\mathrm{O}(p,q) = \{A \in \mathrm{GL}(n,\mathbb{R}); \quad A'E_{p,q}A = E_{p,q}.\}$$

We have $\mathrm{O}(q,q) = \mathrm{O}(q,p)$. Hence we can restrict to $p \geq q$. We write also

$$\mathrm{O}(p,\mathbb{R}) := \mathrm{O}(p,0).$$

These are compact groups.

2.1 Proposition. *The groups*

$$O(p,q), \quad p \geq q \geq 0, \quad p \geq 2,$$

are semisimple up to $(p,q) = (2,0)$. *Up to* $(p,q) = (2,0),(4,0),(2,2)$ *they are simple.*

The groups $O(p,q)$ are not connected. They contain the groups

$$SO(p,q) = O(p,q) \cap SL(p+q,\mathbb{R})$$

as open and closed subgroup of index two. The SO groups still need not to be connected

2.2 Proposition. *The groups*

$$SO(p,q), \quad p \geq q \geq 1, \quad p \geq 2,$$

are not connected. They consist of two connected components. The groups

$$SO(p,\mathbb{R}) := SO(p,0), \quad p \geq 2,$$

are connected.

Unitary groups

Let now V be a complex vector space and $\langle\cdot,\cdot\rangle$ be a hermitian form. This means that it is $\mathbb{C}$ linear in the second variable and that $\langle a,b\rangle = \overline{\langle b,a\rangle}$. As in the real case there exists a basis $e_1,\dots e_n$ such that

$$\langle e_i, e_j\rangle = E_{p,q}$$

with unique (p,q) that is called the signature too. The unitary group $U(V,\langle\cdot,\cdot\rangle)$ is the subgroup of $GL(V)$ that preserves the hermitian form. It is isomorphic to

$$U(p,q) = \{A \in GL(n,\mathbb{C}); \quad \bar{A}' E_{p,q} A = E_{p,q}.\}$$

Again we have $U(p,q) = U(q,p)$. The groups

$$U(n) = U(n,0)$$

are compact. As in the orthogonal case we define

$$SU(p,q) = U(p,q) \cap SL(p+q,\mathbb{C}), \quad SU(n) = SU(n,0).$$

2.3 Proposition. *The groups*

$$SU(p,q), \quad p \geq q \geq 1,$$

are connected and simple.

Symplectic groups

Let V be a real vector space and let now $(\cdot,\cdot)$ be a non degenerated alternating pairing. The latter means that $(a,b) = -(b,a)$. Then the dimension of V is even, $2n$, and there exists a basis such that

$$(e_i, e_j) = I_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

The symplectic group $\mathrm{Sp}(V)$ is the subgroup of $\mathrm{GL}(V)$ that preserves the alternating form. Its matrix presentation is

$$\mathrm{Sp}(2n,\mathbb{R}) = \{M \in \mathrm{GL}(2n,\mathbb{R}); \quad M' I_n M = I_n\}.$$

Clearly symplectic matrices have determinant ± 1. Not so clear is that they always have determinant 1. Even more, the following stronger result holds.

2.4 Proposition. *The symplectic groups* $\mathrm{Sp}(n,\mathbb{R})$, $n \geq 1$, *are connected and simple.*

Variants of symplectic groups

The hermitian symplectic group is

$$\mathrm{Sp}_{\mathrm{herm}}(2n) = \{M \in \mathrm{GL}(2n,\mathbb{C}); \quad \bar{M}' I_n M = I_n\}.$$

Since the field of complex numbers is commutative, we can write this equation as

$$\bar{M}' H M = H, \quad H = \mathrm{i} I_n.$$

The matrix H is hermitian and its signature is (n,n). Hence we have the following result.

2.5 Proposition. *The groups* $\mathrm{Sp}_{\mathrm{herm}}(2n)$ *and* $\mathrm{U}(n,n)$ *are isomorphic.*

The groups $\mathrm{U}(n,n)$ are not simple since they have a big center. Therefore it is better to consider $\mathrm{SU}(n,n)$. The corresponding group is

$$\mathrm{SSp}_{\mathrm{herm}}(2n) := \mathrm{Sp}_{\mathrm{herm}}(2n) \cap \mathrm{SL}(2n,\mathbb{C}).$$

A straight forward calculation shows the following result.

2.6 Remark.

$$\mathrm{SSp}_{\mathrm{herm}}(2) = \mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}).$$

The quaternionic symplectic group is

$$\mathrm{Sp}_{\mathrm{quat}}(2n) = \{M \in \mathrm{GL}(2n, \mathbb{H}); \quad \bar{M}' I_n M = I_n\}.$$

Here we make use of the field of quaternions Let's recall the basic properties of the field $\mathbb{H}$ of Hamilton quaternions: We denote the standard basis of this field by $1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3$,

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathrm{i}_1 + \mathbb{R}\mathrm{i}_2 + \mathbb{R}\mathrm{i}_3.$$

The defining relations are

$$\begin{aligned} &\mathrm{i}_1^2 = \mathrm{i}_2^2 = \mathrm{i}_3^2 = -1, \\ &\mathrm{i}_1\mathrm{i}_2 = -\mathrm{i}_2\mathrm{i}_1 = \mathrm{i}_3, \ \mathrm{i}_2\mathrm{i}_3 = -\mathrm{i}_3\mathrm{i}_2 = \mathrm{i}_1, \ \mathrm{i}_3\mathrm{i}_1 = -\mathrm{i}_1\mathrm{i}_3 = \mathrm{i}_2. \end{aligned}$$

It is known that this defines a skew field. The conjugate $\bar{x}$ of a quaternion $x = x_0 + x_1\mathrm{i}_1 + x_2\mathrm{i}_2 + x_3\mathrm{i}_3$ is

$$\bar{x} := x_0 - x_1\mathrm{i}_1 - x_2\mathrm{i}_2 - x_3\mathrm{i}_3.$$

One has $\overline{xy} = \bar{y}\bar{x}$ and $x\bar{x} = x_0^2 + \cdots + x_3^2$. The group $\mathrm{GL}(n, \mathbb{H})$ is defined as in the commutative case. There is a ring homomorphism

$$\alpha : \mathbb{H} \longrightarrow \mathbb{C}^{2\times 2}, \quad \alpha(x) = \begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix}.$$

(This shows that $M_2(\mathbb{C})$ is the complexification of $\mathbb{H}$. This will be treated later). A simple computation shows

$$\alpha(\bar{x}) = \overline{\alpha(x)}'.$$

The image of α consists of all complex matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

They can be characterised by the property

$$I_1^{-1} A I_1 = \bar{A}.$$

We extend this embedding to a map

$$\alpha : \mathrm{GL}(n, \mathbb{H}) \longrightarrow \mathrm{GL}(2n, \mathbb{C})$$

The image consists of all matrices A such that

$$\begin{pmatrix} I_1 & & \\ & \ddots & \\ & & I_1 \end{pmatrix}^{-1} A \begin{pmatrix} I_1 & & \\ & \ddots & \\ & & I_1 \end{pmatrix} = \bar{A}.$$

by replacing each entry by the corresponding 2×2-matrix. The map α induces a homomorphism

$$\alpha : \mathrm{Sp}_{\mathrm{quat}}(2n) \longrightarrow \mathrm{GL}(4n, \mathbb{C}).$$

The image is contained in the intersection of the hermitian symplectic group and the special linear group. Hence we get the following conditions for the image

$$\det(M) = 1,\ \bar{M}' I_{2n} M = I_{2n},\ \begin{pmatrix} I_1 & & \\ & \ddots & \\ & & I_1 \end{pmatrix}^{-1} M \begin{pmatrix} I_1 & & \\ & \ddots & \\ & & I_1 \end{pmatrix} = \bar{M}.$$

We insert the third condition into the first one. In this way we see that the image is characterized by

$$\det M = 1,\ \bar{M}' I_{2n} M = I_{2n},\ M'SM = S,\ S = \begin{pmatrix} & & I_1 & & \\ & & & \ddots & \\ & & & & I_1 \\ I_1' & & & & \\ & \ddots & & & \\ & & I_1' & & \end{pmatrix}.$$

We can define an isomorphic group by applying $M \mapsto NMN^{-1}$. We demand that $N \in \mathrm{Sp}_{\mathrm{herm}}(4n)$. Then we just get a change of S to $T = N'SN$. We apply this principle two times. For sake of simplicity we restrict to $n = 2$. In the first step we transform S to

$$\begin{pmatrix} 0 & I_1 & & \\ I_1' & 0 & & \\ & & 0 & I_1 \\ & & I_1' & 0 \end{pmatrix}.$$

Here we can use

$$N = \begin{pmatrix} A' & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the next step we are reduced to transform

$$\begin{pmatrix} 0 & I_1 \\ I_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here we use

$$\begin{pmatrix} \bar{A}' & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & \\ & \mathrm{i} & & \\ & & 1 & \\ & & & \mathrm{i} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

This transforms S finally to the unit matrix. Hence we see that the group $\mathrm{Sp}(2n, \mathbb{H})$ is embedded into the group

$$\mathrm{Sp}_{\mathrm{herm}}(4n) \cap \mathrm{SO}(4n, \mathbb{C}).$$

Here the first time orthogonal groups with complex coefficients appear:

$$\mathrm{O}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}), \quad A'A = E\}, \quad \mathrm{SO}(n, \mathbb{C}) = \mathrm{O}(n, \mathbb{C}) \cap \mathrm{SL}(n, \mathbb{C}).$$

Now we introduce the group

$$\mathrm{SO}^*(2n) = \mathrm{Sp}_{\mathrm{herm}}(2n) \cap \mathrm{SO}(2n, \mathbb{C}).$$

2.7 Proposition. *The groups* $\mathrm{SO}^*(2n)$, $n \geq 3$, *are connected and simple.*

We omit a proof. A dimension comparison now shows

2.8 Proposition. *The groups* $\mathrm{Sp}_{\mathrm{quat}}(2n)$, $\mathrm{SO}^*(4n)$, $n \geq 2$ *are isomorphic.*

3. Exceptional isogenies

In the following the group $\mathrm{SL}(2, \mathbb{R})$ will occur several times. It is worth while to mention that it is a symplectic group.

3.1 Lemma. *One has*

$$\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R}).$$

We consider the space V of real symmetric 2×2-matrices. There exists a symmetric bilinear form with the property $(X, X) = -2 \det X$. Just set $(X, Y) = \det(X) + \det(Y) - \det(X + Y)$. The signature of this form is $(2, 1)$. Hence we have that $\mathrm{O}(V, (\cdot, \cdot))$ is isomorphic to $\mathrm{O}(2, 1)$. Let $A \in \mathrm{SL}(2, \mathbb{R})$. We get a linear map

$$V \longrightarrow V, \quad X \longmapsto AXA'.$$

This map preserves the determinant. Hence we get a homomorphism

$$\mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{O}(2, 1).$$

3.2 Proposition. *The groups* $\mathrm{SL}(2, \mathbb{R})$ *and* $\mathrm{O}(2, 1)$ *are isogenous.*

We treat a variant of this construction. We consider the space V of all hermitian 2×2-matrices

$$H = \begin{pmatrix} h_0 & h_1 \\ \bar{h}_1 & h_2 \end{pmatrix}.$$

We identify V with $\mathbb{R}^4$ through

$$H \longmapsto \Big(\frac{h_0 + h_2}{2}, \frac{h_0 - h_2}{2}, \operatorname{Re} h_1, \operatorname{Im} h_1\Big).$$

Then we have

$$\det H = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The group $\mathrm{SL}(2, \mathbb{C})$ acts on V through $(A, H) \longmapsto AH\bar{A}'$ It preserves the determinant. Hence we obtain a Lorentz transformation. We get a homomorphism

$$\mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(3, 1).$$

The image is an open subgroup $\mathrm{SO}^+(3, 1)$ of index two. Each element of the image has two inverse images which differ by the sign.

3.3 Proposition. *The groups* $\mathrm{SL}(2, \mathbb{C})$ *and* $\mathrm{SO}(3, 1)$ *are isogenous.*

We restrict the homomorphism $\mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(3, 1)$ to the unitary group $\mathrm{SU}(2)$. This group acts on the subspace of V defined by $\mathrm{tr}(H) = 0$. This is a three dimensional space and the scalar product is definit on it. Hence we get a homomorphism

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3, \mathbb{R}).$$

It can shown that it is surjective and every element has two inverse images that differ by the sign.

3.4 Proposition. *The groups* $\mathrm{SU}(2)$ *and* $\mathrm{SO}(3, \mathbb{R})$ *are isogenous.*

Now we denote by V the space of all real 2×2-matrices. Its dimension is 4. Here the determinant (or its negative) define a quadratic form of signature $(2, 2)$. Hence now we have that $\mathrm{O}(V, (\cdot, \cdot))$ is isomorphic to $\mathrm{O}(2, 2)$. Let $A, B \in \mathrm{SL}(2, \mathbb{R})$. Then we get a linear map

$$V \longrightarrow V, \quad X \longmapsto AXB'.$$

This defines a homomorphism

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{O}(2, 2).$$

3.5 Proposition. *The groups* $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ *and* $\mathrm{O}(2, 2)$ *are isogenous.*

Next we consider $\mathrm{O}(2,3)$. Here we use the vector space V of all real matrices of the form

$$X = \begin{pmatrix} x_5 & -x_3 & 0 & -x_1 \\ x_4 & -x_5 & x_1 & 0 \\ 0 & -x_2 & x_5 & x_4 \\ x_2 & 0 & -x_3 & -x_5 \end{pmatrix}.$$

Here we use the quadratic form

$$2(X,X) = \mathrm{tr}(X^2) = 2(x_1x_2 + x_3x_4 - x_5^2).$$

The signature is $(2,3)$. A straight forward check shows that $\mathrm{Sp}(4,\mathbb{R})$ acts on V through

$$X \longmapsto MXM^{-1}.$$

This gives a homomorphism $\mathrm{Sp}(4,\mathbb{R}) \longmapsto \mathrm{O}(2,3)$.

3.6 Proposition. *The groups* $\mathrm{Sp}(4,\mathbb{R})$ *and* $\mathrm{SO}(2,3)$ *are isogenous.*

Next we treat $\mathrm{O}(2,4)$. Here we use the real vector space V of all complex matrices of the form

$$X = \begin{pmatrix} x_0 & -x_3 & 0 & -x_1 \\ x_4 & -\bar{x}_0 & x_1 & 0 \\ 0 & -x_2 & x_0 & x_4 \\ x_2 & 0 & -x_3 & -\bar{x}_0 \end{pmatrix}.$$

Here we use the quadratic form

$$2(X,X) = \mathrm{tr}(X\bar{X}) = 2(x_1x_2 + x_3x_4 - x_5^2 - x_6^2).$$

The group $\mathrm{Sp}_{\mathrm{herm}}(4) \cap \mathrm{SL}(4,\mathbb{C})$ acts on V through

$$X \longmapsto MX\bar{M}^{-1}.$$

This gives a homomorphism

$$\mathrm{Sp}_{\mathrm{herm}}(4) \cap \mathrm{SL}(4,\mathbb{C}) \longrightarrow \mathrm{O}(2,4).$$

As we know, $\mathrm{Sp}_{\mathrm{herm}}(4) \cap \mathrm{SL}(4,\mathbb{C})$ is isomorphic to $\mathrm{U}(2,2)$.

3.7 Proposition. *The groups* $\mathrm{SU}(2,2)$ *and* $\mathrm{O}(2,4)$ *are isogenous.*

There is a similar exceptional isogeny.

3.8 Proposition. *The groups* $\mathrm{Sp}_{\mathrm{quat}}(4)$ *and* $\mathrm{O}(2,6)$ *are isogenous.*

We cannot give a proof here. We just mention that there exists an isomorphism

$$\mathrm{Sp}_{\mathrm{quat}}(4)/\pm E \cong \mathrm{SO}(2,6)/\pm E.$$

But this isomorphism does not lift to a homomorphism from $\mathrm{Sp}_{\mathrm{quat}}(4)$ into $\mathrm{SO}(2,6)$.

An interesting group is $\mathrm{SO}(4,\mathbb{R})$. We mentioned already that it is semisimple but not simple. We give some details here. We start with the map $\alpha : \mathbb{H} \to \mathbb{C}^{2\times 2}$ that we introduced in Sect. 2. We restrict it to the multiplicative group $\mathbb{H}(1)$ of quaternions of norm 1. A straight forward calculation shows that

$$\mathbb{H}(1) \xrightarrow{\sim} \mathrm{SU}(2)$$

is an isomorphism. Next we consider the group $\mathbb{H}(1)\times\mathbb{H}(1)$. It acts on $\mathbb{H}$ through

$$(\mathbb{H}(1)\times\mathbb{H}(1))\times\mathbb{H} \longrightarrow \mathbb{H}, \quad ((a,b),x) \longmapsto ax\bar{b}.$$

It preserves the norm. Hence, if we identify $\mathbb{H}$ with $\mathbb{R}^4$, we get a homomorphism

$$\mathbb{H}(1)\times\mathbb{H}(1) \longrightarrow \mathrm{SO}(4).$$

It can be shown that this surjective and that the kernel is $\pm(1,1)$. In this way we see

3.9 Proposition. *The groups* $\mathrm{SO}(4)$ *and* $\mathrm{SU}(2)\times\mathrm{SU}(2)$ *are isogenous.*

4. Lattices

4.1 Definition. *A lattice in a unimodular group* G *is a discrete subgroup* $\Gamma \subset G$ *such that* $\Gamma\backslash G$ *has finite volume.*

Hence we can talk about lattices in semisimple Lie groups.

Two subgroups G_1, G_2 of a group G are called *commensurable* if the intersection $G_1 \cap G_2$ has finite index in both, G_1, G_2. it can be shown that this is an equivalence relations. Hence the subgroups of a group can be grouped into equivalence classes of commensurable groups.

4.2 Remark. *Let* $\Gamma \subset G$ *be a lattice in a unimodular group. Then every commensurable group is a lattice too.*

4.3 Definition. *A subgroup Γ in a semisimple Lie group G is **reducible** if there exist simple Lie groups $G_1, \dots G_n$ of positive dimension and subgroups $\Gamma_i \subset G_i$ and a surjective continuous homomorphism $G_1 \times \dots \times G_n \to G^0$ with finite kernel such that the image of $\Gamma_1 \times \dots \times \Gamma_n$ is contained in Γ and has finite index there.*

The group Γ is called irreducible if it is not reducible. Every lattice in a simple group is irreducible.

5. Linear algebraic groups

We need the notion of a (linear) algebraic group. This is a subgroup of $\mathrm{GL}(N, \mathbb{C})$ with certain properties. We will consider polynomials on $\mathbb{C}^{N \times N}$. They form a ring $\mathbb{C}[X_{ij}]$. For example $\det X$ is a polynomial. Let $L \subset \mathbb{C}$ be a subfield of the field of complex numbers.

5.1 Definition. *A subgroup of $\mathbf{G} \subset \mathrm{GL}(N, \mathbb{C})$ is called linear algebraic defined over L if there exists a finite number of polynomials $P_1, \dots, P_m \in L[X_{ij}]$ with coefficients in L such that*

$$\mathbf{G} = \{g \in \mathrm{GL}(N, \mathbb{C}); \quad P_1(g) = \ldots = P_m(g) = 0\}.$$

So to an linear algebraic group $\mathbf{G}$ belongs an embedding into $\mathrm{GL}(N, \mathbb{C})$ for a suitable N. In particularly, for any subring $A \subset \mathbb{C}$ with unit one can define the group

$$\mathbf{G}(A) = \mathbf{G} \cap \mathrm{GL}(N, A).$$

The category of linear algebraic groups over a field

We define the category of linear algebraic groups defined over the field L. Objects are linear algebraic groups $\mathbf{G} \subset \mathrm{GL}(N, \mathbb{C})$. We introduce the notion of a regular function on a linear algebraic group. A function on $\mathrm{GL}(n, \mathbb{C})$ is called *regular* (over L) if it is the product of a polynomial in the X_{ij} with coefficients in L and an integral power (which my be negative) of $\det X$. Let now $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic subgroup over L. Then we define the notion of a regular function on $\mathbf{G}$ defined over L to be the restriction of a regular function on $\mathrm{GL}(N, \mathbb{Z})$ defined over L. We denote by $L[\mathbf{G}]$ the set of all regular functions. This is a commutative $\mathbb{C}$-algebra. A morphism of linear algebraic groups over L $f : \mathbf{G} \to \mathbf{G}'$ ($\mathbf{G} \subset \mathrm{GL}(N, \mathbb{C})$, $\mathbf{G}' \subset \mathrm{GL}(N', \mathbb{C})$) is a homomorphism of groups such that the components are given by regular functions over L. The morphism f is called an isomorphism of linear groups defined over L if it is an isomorphism of groups and if f^{-1} exists and is a morphism over L as well.

Let $\mathbf{G} \subset \mathrm{GL}(N, \mathbb{C})$ be a linear algebraic subgroup (over L). A subgroup subgroup $\mathbf{H} \subset \mathbf{G}$ is called an algebraic subgroup defined over L if it is a linear algebraic subgroup (over L) of $\mathrm{GL}(N, \mathbb{C})$. Then the naturali inclusion $\mathbf{H} \hookrightarrow \mathbf{G}$ is a homomorphism defined over L.

A linear algebraic subgroup $\mathbf{G}$ that is defined over L is also defined over every field L', $L \subset L' \subset \mathbb{C}$.

The group $\mathbf{G} = \mathrm{GL}(N, \mathbb{C})$ is an algebraic subgroup (defined over $\mathbb{Q}$). In the case $N = 1$ this means that $\mathbf{G}_m = \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$ is linear algebraic over $\mathbb{Q}$.

5.2 Definition. *An algebraic subgroup* $\mathbf{T}$ *of a linear algebraic subgroup, both defined over* L *is called an* L*-split torus if* $\mathbf{T}$ *is isomorphic to* $\mathbf{G}_m^r$ *in the category of linear algabraic groups over* L.

There is a maximal r with this property. This r is called the L-rank of $\mathbf{G}$

5.3 Definition. *A linear algebraic group* $\mathbf{G}$ *over a subfield of* $\mathbb{R}$ *is called semisimple if* $\mathbf{G}(\mathbb{R})$ *is a semisimple Lie group.*

Not every semisimple Lie group arises in this way. Counter examples are the metaplectic groups.

5.4 Theorem. *Let* $\mathbf{G} \subset \mathrm{GL}(N, \mathbb{C})$ *be a semisimple linear algebraic group defined over* $\mathbb{Q}$. *Then* $\mathbf{G}(\mathbb{Z})$ *is a lattice in* $G = \mathbf{G}(\mathbb{R})$.

This is a consequence of the so-called reduction theory. Borel, Armand (1969), Introduction aux groupes arithmtiques, In the special cases $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ this is the Minkowski reduction theory. We will treat this and Siegel's extension to $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{Sp}(n, \mathbb{Z})$ in detail in Chapter II.

5.5 Definition. *Let* G *be semisimple Lie group. An irreducible subgroup* $\Gamma \subset G$ *is called* ***arithmetic*** *if there exists a semisimple linear algebraic group* $\mathbf{G}$ *defined over* $\mathbb{Q}$ *such that there exists an isomorphism of Lie groups* $G \cong \mathbf{G}(\mathbb{R})$ *such that the image of* Γ *is commensurable with* $\mathbf{G}(\mathbb{Z}) \cap G$.

There is a generalization of the notion of arithmetic group that includes also non irreducible Γ. We don't need this.

5.6 Theorem. *Arithmetic subgroups are lattices.*

The famous arithmeticy theorem of Margulis states.

5.7 Theorem. *Let* $\mathbf{G}$ *be a linear algebraic group defined over* $\mathbb{Q}$ *such that the* $\mathbb{R}$*-rank is* ≥ 2. *Assume that* $G = \mathbf{G}(\mathbb{R})$ *has no compact factor different from* $\{e\}$. *Then every irreducible lattice in* G *is arithmetic.*

1975, Margulis obtained the fields medal for his groundbreaking research on lattices.

6. Discontinuous group action

Let X be a locally compact Hausdorff space with countable basis of topology. Let $f : \Gamma \to \mathrm{Top}(X)$ be a homomorphism of Γ into the group of topological self-mappings of X. We say that Γ acts topologically on X. We also can describe the action by means of the map

$$\Gamma \times X \longrightarrow X, \quad (\gamma, x) \longmapsto \gamma x := f(\gamma)(x).$$

6.1 Definition. *Let Γ be a group that acts topologically on X. This action is called **discontinuous** if for any two compact subsets $K, K' \subset X$ the set*

$$\{\gamma \in \Gamma; \quad \gamma(K) \cap K' \neq \emptyset\}$$

is finite.

Since we can consider $K \cup K'$, it is sufficient to restrict to the case $K = K'$. We call two points x, y of X equivalent if there exists $\gamma \in \Gamma$ such that $y = \gamma(x)$. We denote the quotient (set of equivalence classes) by $\Gamma\backslash X$ and equip it with the quotient topology.

6.2 Lemma. *Assume that Γ acts discontinuously on X. Then $\Gamma\backslash X$ is a locally compact Hausdorff space with countable basis of the topology. The quotient map $X \to \Gamma\backslash X$ is continuous and open. For each $a \in X$ the stabilizer*

$$\Gamma_a = \{\gamma \in \Gamma; \ \gamma(a) = a\}$$

is a finite group. The map

$$\Gamma_a\backslash X \longrightarrow \Gamma\backslash X$$

maps a small open neighborhood of the image of a in $\Gamma_a\backslash X$ topologically to an open neighborhood of the image of a in $\Gamma\backslash X$.

6.3 Lemma. *Let G be a locally compact group, let $K \subset G$ be a compact subgroup and let $\Gamma \subset G$ be a discrete subgroup. Then Γ acts discontinuously on G/K (through multiplication from the left).*

6.4 Proposition. *Let G be a locally compact group and X a locally compact Hausdorff space, both with countable basis of the topology. Let $G \to \mathrm{Top}(X)$ be a homomorphism such that $G \times X \longrightarrow X$ is continuous. Assume that $a \in X$ is a point such that the natural map*

$$H\backslash G \longrightarrow X, \quad H = \{g \in G; \ g(a) = a\},$$

is bijective. Then it is topological.

It is clear that $H\backslash G \longrightarrow X$ is continuous, the statement is that the converse map is continuous too. This is equivalent to the fact that $G \to X$, $g \mapsto g(a)$, is open. The proof uses the Baire category theorem (Helgason, differential geometry and symmetric spaces, Chap. II, Theorem 3.2).

Quotient measure

Let (X, dx) be a Radon measure and Γ a groups that acts discontinuously on X. We assume that dx is invariant under Γ. Then one can define a quotient measure $\Gamma\backslash X$. The defining formula is

$$\int_X f(x)dx = \int_{\Gamma\backslash X} \sum_{\gamma\in\Gamma} f(\gamma x)dx \qquad (f \in \mathcal{C}_c(X)).$$

In the special case where G is a unimodular group and Γ is a discrete subgroup, this coincides with the quotient Haar measure that we defined above. Another way to define is to use fundamental domains.

6.5 Definition. *Let Γ be a group. that acts discontinuously on X. A Borel subset $F \subset X$ is called a **fundamental set** if X is the union of all translates $\gamma(F)$, $\gamma \in \Gamma$.*

6.6 Definition. *Let (X, dx) be a Radon measure and let Γ be a group that acts discontinuously on X. A **fundamental domain** is a fundamental set a such that in addition the set of all $x \in F$ which are equivalent so some $y \in F$, $y \neq x$, is a zero set.*

One can show that any fundamental set contains a fundamental domain. Let $f : \Gamma\backslash X \to \mathbb{C}$. We can pull it back to a function $F : X \to \mathbb{C}$ with the property $F(\gamma x) = F(x)$. Sometimes we identify f and F and use the same letter for them.

6.7 Proposition. *Let (X, dx) be a Radon measure and let Γ be a discontinuous group acting on X. Let $\mathcal{F}$ be a fundamental set. We assume that dx is invariant under Γ. Let f be an integrable function (X, dx) and let F the pull back on X. Then F is integrable along $\mathcal{F}$ and*

$$\int_X f(x)dx = \int_{\mathcal{F}} F(x)dx.$$

It follows

$$\mathrm{vol}(\Gamma\backslash X) \leq \mathrm{vol}(\mathcal{F})$$

for each fundamental set. Equality holds for fundamental domains.

7. Maximal compact subgroups

A subgroup $K \subset G$ of a topological group is called maximal compact if it is compact and if it equals every compact subgroup K' such that $K \subset K'$.

7.1 Proposition. *Any semisimple group G admits a maximally compact subgroup K. Two maximal compact subgroups are conjugate. Every compact subgroup is contained in a maximal compact subgroup.*

7.2 Proposition. *Let K be maximal compact in the semisimple group G. Then there exists a closed connected subgroup $P \subset G$ such that*

$$P \times K \longrightarrow G, \quad (p, k) \longmapsto pk,$$

is a diffeomorphism and such that the Haar measure on G corresponds to the direct product of the Haar measures dp and dk on $P \times K$.

We consider the coset space G/K. We know that it carries an essentially unique Radon measure that is invariant under left translation under K. The group Γ acts discontinuously on this space and we can consider the quotient measure on $\Gamma\backslash G/K$.

7.3 Remark. *We have*

$$\operatorname{vol}(\Gamma\backslash G) < \infty \Longleftrightarrow \operatorname{vol}(\Gamma\backslash G/K) < \infty.$$

8. Bounded symmetric domains

The famous Riemann mapping theorem states that every simply connected domain in $\mathbb{C}$ that is different form $\mathbb{C}$ is biholomorphic equivalent to the unit disc. The straightforward generalization to many complex variables is false. There are much more restrictive conditions necessary to get a generalization.

8.1 Definition. *A bounded symmetric domain is a bounded connected open subset D of $\mathbb{C}^n$ such that for every point $a \in D$ there exists a biholomorphic mapping of D onto itself that has a as isolated fixed point.*

We denote by $\operatorname{Bihol}(D)$ the group of all biholomorphic self mappings of D.

8.2 Proposition. *Let D be a bounded symmetric domain. The group* $\mathrm{Bihol}(D)$ *can be equipped with a structure as Lie group. A sequence converges in the topology of this Lie group if and only if it converges uniformly on compact subsets. Two bounded symmetric domains D_1, D_2 are biholomorphic if and only if the groups* $\mathrm{Bihol}(D_1)^0$, $\mathrm{Bihol}(D_2)^0$ *are isomorphic Lie groups.*

8.3 Proposition. *Let D be a bounded symmetric domain. The group* $G = \mathrm{Bihol}(D)$ *is semisimple. Its connected component is of adjoint type. The stabilizer of a point is a maximal compact subgroup K. The group acts transitively on D. The map*

$$G/K \longrightarrow D$$

is a diffeomorphism.

We call a semisimple group of hermitian type if its adjoint group is isomorphic to the connected component of the group of biholomorphic self mappings of a bounded symmetric domain.

8.4 Theorem. *A simple group G is of Hermitian type if and only if its maximal compact subgroups are not semi-simple.*

Irreducibility

The product $D = D_1 \times D_2$ of hermitian symmetric domains is a hermitian symmetric domain. The group $\mathrm{Bihol}(D_1) \times \mathrm{Bihol}(D_2)$ is naturally embedded into $\mathrm{Bihol}(D)$. It can be shown that this is a subgroup of finite index. A symmetric domain D is called irreducible if it is not biholomorphic equivalent to such a product. A hermitian symmetric domain is irreducible if and only if $\mathrm{Bihol}(D)$ is a simple Lie group.

8.5 Theorem. *Simple groups of hermitian type.*

Type (Siegel)	Semisimple Lie Group	maximal compact		dimension of domain
I_{pq}	$\mathrm{SU}(p,q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$q \geq p \geq 1$	pq
II_n	$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$	$n \geq 3$	$n(n-1)/2$
III_n	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	$n \geq 1$	$n(n+1)/2$
IV_n	$\mathrm{SO}(2,n)$	$\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(n))$	$n \geq 1, n \neq 2$	n

Every simple group of hermitian domain is isogenic to at least one of this list, except to two exceptional groups. The complex dimensions of the two exceptional domains are 16 *and* 27*.*

9. Examples of bounded hermitian domains

The most classical example comes from the group $\mathrm{SL}(2,\mathbb{R})$. The maximal compact subgroup is $\mathrm{SO}(2)$ which is abelian, hence not semisimple. The standard model for the domain is the upper half plane $\mathcal{H} = \{z;\ \operatorname{Im} z > 0\}$. This is not bounded but it is biholomorphic equivalent to a bounded domain, namely the unit disc. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{H}$ through Möbius transformations

$$z \longmapsto (az+b)(cz+d)^{-1}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group $\mathrm{SO}(2,\mathbb{R})$ is the stabilizer of the point i. So we have the bijective map that is topological in fact

$$\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) \xrightarrow{\sim} \mathcal{H}, \quad M\,\mathrm{SO}(2,\mathbb{R}) \longmapsto M(\mathrm{i}).$$

The basic arithmetic group is the elliptic modular group $\mathrm{SL}(2,\mathbb{R})$.

The next classical example is the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. We already know $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. The generalized upper half plane, often called the Siegel upper half plane, is a subdomain of the vector space $\mathcal{Z}_n$ of symmetric complex $n \times n$-matrices.

$$\mathcal{H}_n = \{Z = X + \mathrm{i}Y \in \mathcal{Z}_n; \quad Y > 0\}.$$

Of course $Y > 0$ means that Y is positive definit. One has to prove that the group $\mathrm{Sp}(2n,\mathbb{R})$ acts on $\mathcal{H}_n$ through

$$MZ = (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

As distinguished point one takes $\mathrm{i}E_n$. The stabilizer of this point is

$$\mathrm{O}(2n,\mathbb{R}) \cap \mathrm{Sp}(2n,\mathbb{R}).$$

This group os isomorphic to the unitary group $\mathrm{U}(n)$,

$$\mathrm{O}(2n,\mathbb{R}) \cap \mathrm{Sp}(2n,\mathbb{R}) \xrightarrow{\sim} \mathrm{U}(n), \quad M \longmapsto C\mathrm{i} + D.$$

We identify both groups. Hence we can write

$$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n) \cong \mathcal{H}_n.$$

10. The complexification of Lie groups

Besides the notion of a real Lie group one also can introduce the notion of a complex Lie group. This is a group that has been equipped with the structure of a a complex manifold such that multiplication and inversion are complex analytic. A homomorphism of complex Lie groups means a homomorphism which is als a complex analytic map. Every complex Lie group G can be considered also as a usual (real) Lie group. We denote this Lie group by $G^{\mathbb{R}}$. The assignment $G \mapsto G^{\mathbb{R}}$ is a functor. Examples for complex Lie groups are $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, the complex symplectic group

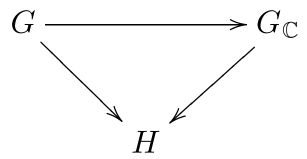
$$\mathrm{Sp}(2n, \mathbb{C}) = \{M \in \mathrm{GL}(2n, \mathbb{C}); \quad M' I_n M = I_n\}.$$

Finally the complex orthogonal group

$$\mathrm{O}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}); \quad A'A = E$$

is a complex Lie group.

10.1 Definition. *A universal complexification of a (real) Lie group G is a complex Lie group $G_{\mathbb{C}}$ together with a homomorphism of Lie groups $\iota : G \to G_{\mathbb{C}}$ such that for each complex Lie group H and a homomorphism of Lie groups $G \to H$ a unique homomorphism of complex Lie groups $G_{\mathbb{C}} \to H$ exists such that the diagram*



is commutative.

Of course the pair $(G_{\mathbb{C}}, \iota)$ is uniquely determined up to a unique isomorphism. One can show that a universal complexification exists. But the homomorphism ι is not always injective. But in all cases of our interest this will be the case.

It can be proved that ι is injective for compact groups. Many examples where ι is injective comes from algebraic groups.

10.2 Proposition. *Let $\mathbf{G}$ be a linear algebraic group defined over $\mathbb{R}$ then $\mathbf{G}(\mathbb{C})$ is a universal complexification of $\mathbf{G}(\mathbb{R})$.*

From this observation we get some examples of complexifications:

$$\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C}), \quad \mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{Sp}(2n, \mathbb{C}), \quad \mathrm{SO}(p, q) \subset \mathrm{SO}(p+q, \mathbb{C}).$$

10.3 Proposition. *Let $\mathbf{G} \to \mathbf{H}$ be a homomorphism of linear algebraic groups, then the pull-back of a regular function on $\mathbf{H}$ is regular on $\mathbf{G}$. This induces a homomorphism*

$$\mathbb{C}[\mathbf{H}] \longrightarrow \mathbb{C}[\mathbf{G}].$$

We omit the proof of this not trivial result. We just mention that it is the basis for the algebraic geometric approach to linear algebraic groups. □

Next we formulate a basic result about complexifications.

10.4 Proposition. *Let* $\mathbf{G}$ *be a linear algebraic group over* $\mathbb{C}$*. Assume that a continuous homomorphism* $\tau : \mathbf{G}(\mathbb{C}) \to \mathbf{G}(\mathbb{C})$ *with the following properties is given.*

1) $\tau^2 = \mathrm{id}$ *(so* τ *is an involution.*
2) *The function* $\overline{\tau(g)}$ *is regular.*

Then there exists a linear algebraic group $\mathbf{H}$ *defined over* $\mathbb{R}$ *together with an isomorphism of algebraic groups* $\mathbf{H} \to \mathbf{G}$ *(over* $\mathbb{C}$*) such that* $\mathbf{H}(\mathbb{R})$ *corresponds to the group of all*

$$g \in \mathbf{G}(\mathbb{C}); \quad \tau(g) = g.$$

The trivial example is the case of a $\mathbf{G}$ that is already defined of $\mathbb{R}$. Then we can take $\tau(g) = \bar{g}$ and we obtain the original $\mathbb{R}$-structure.

We give another example. Let $\mathbf{G} = \mathrm{GL}(n, \mathbb{C})$, $n = p + q$ and

$$\tau(g) = E_{pq} \bar{g}'^{-1} E_{pq}.$$

Then the proposition gives us the existence of an algebraic group $\mathbf{G}$ over $\mathbb{R}$ that is isomorphic to $\mathrm{GL}(n, \mathbb{C})$ as linear algebraic group over $\mathbb{C}$ and such that

$$\mathbf{G}(\mathbb{R}) \cong \mathrm{U}(p, q).$$

10.5 Lemma. *The complexification of* $\mathrm{U}(p, q)$ *is* $\mathrm{GL}(n, \mathbb{C})$*. Similarly, the complexification of* $\mathrm{SU}(n)$ *is* $\mathrm{SL}(2, \mathbb{C})$*.*

Instead of a general proof of Proposition 1.10.4 we look at this example. First we treat the simplest case $n = 1$ and $p = 1$, $q = 0$. We have to construct an algebraic group over $\mathbb{R}$ that is isomorphic to $\mathbb{C}^*$ ober $\mathbb{C}$ and such that $\mathbf{G}(\mathbb{R})$ corresponds to $\mathrm{U}(1) = S^1$. We claim that $\mathbf{G} = \mathrm{SO}(2, \mathbb{C})$ does the job. Actually

$$\mathrm{SO}(2, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}^*, \quad M \longmapsto c\mathrm{i} + d$$

is an isomorphism defined over $\mathbb{C}$ (but not over $\mathbb{R}$) and $\mathrm{SO}(2\mathbb{R})$ corresponds to S^1.

10.6 Proposition. *The complexifications of the simple groups of hermitian type up to the two exceptional cases are given by*

$I_{p,q}$	$\mathrm{SU}(p, q)$	$\mathrm{SL}(p + q, \mathbb{C})$
II_n	$\mathrm{SO}^*(2n)$	$\mathrm{SO}(2n, \mathbb{C})$
III_n	$\mathrm{Sp}(n, \mathbb{R})$	$\mathrm{Sp}(n, \mathbb{C})$
IV_n	$\mathrm{SO}(2, n)$	$\mathrm{SO}(n + 2, \mathbb{C})$

11. Automorphic forms

Let G be a semisimple group and K be a maximal compact subgroup and let $\Gamma \subset G$ be a lattice. Let $\sigma : K \to \mathrm{GL}(V)$ be a representation (=continuous homomorphism) of K on a finite dimensional complex vector space.

11.1 Definition. *An automorphic form of type σ is a function $f : G \to V$ with the property*

$$f(\gamma x k) = \sigma(k) f(x), \quad \gamma \in \Gamma,\ x \in G,\ k \in K.$$

We are interested in the case where G is of hermitian type. Then there exists a hermitian symmetric domain D together with a homomorphism $G \to \mathrm{Bihol}(D)$ and such that the group K is the stabilizer of some point $a \in D$. So we have a bijection $G/K \to D$. We need a complexification $K \subset K_{\mathbb{C}}$ (which exist for general reasons and which is visible in all our examples). Due to the universal property of the complexification the representation extends to a holomorphic homomorphism

$$\sigma : K_{\mathbb{C}} \longrightarrow \mathrm{GL}(V).$$

11.2 Definition. *A (holomorphic) factor of automorphy with respect to a subgroup $\Gamma \subset G$ and with values in a complex Lie group H is map*

$$J : \Gamma \times D \longrightarrow H$$

with the properties

1) *$J(\gamma, z)$ is holomorphic for fixed γ.*
2) *$J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 z) J(\gamma_2, z)$.*

An example of an automorphy factor with respect to the full Lie group is the Jacobi map $J(g, z)$. Its values are in $\mathrm{GL}(V)$ where V is the vector space that contains D as open subset. But there exist more fundamental automorphy factors.

11.3 Proposition. *There exists an automorphy factor*

$$J_{\mathrm{can}} : G \times D \longrightarrow K_{\mathbb{C}}.$$

with the property

$$J(k, a) = k.$$

We do not discuss the question of uniqueness. We just assume that one J_{can} has been selected.

11.4 Proposition. *For every representation σ of K on a finite dimensional complex vector space there exists an automorphy factor*

$$J : G \times D \longrightarrow K_{\mathbb{C}}.$$

with the property $J(k,a) = \sigma(k)$ for $k \in K$, for example

$$J_\sigma(g,z) := \sigma \circ J_{\mathrm{can}}(g,z).$$

Here σ means the holomorphic extension to $K_{\mathbb{C}}$.

11.5 Lemma. *Let G be a semisimple group of hermitian type, acting on the domain D with distinguished point $a \in D$ and let K be the stabilizer of a distinguished point. Let $\Gamma \subset G$ be a lattice and let $\sigma : K \to \mathrm{GL}(V)$ be a representation on a finite dimensional complex vector space. There is a one-to-one correspondence between the following two spaces of functions*

1) $f : G \to V,\ f(\gamma x k) = \sigma(k) f(x)\ \ (\gamma \in \Gamma,\ x \in G,\ k \in K)$ *(i.e. automorphic forms)*
2) $F : D \to V,\ F(\gamma(z)) = J_\sigma(\gamma, z) F(z),\ (\gamma \in \Gamma,\ z \in D.)$ *The correspondence is given through*

$$F(g) = J_\sigma(g,a)^{-1} f(ga), \quad f(z) = J_\sigma(g^{-1}, z)^{-1} F(g)\ (z = g(i)).$$

The advantage of the description 2) is that one can easily speak of *holomorphic* automorphic forms.

11.6 Preliminary Definition. *Let $G/K \cong D$ be a hermitian symmmetric domain and let $\sigma : K \to \mathrm{GL}(V)$ be a representation on a finite dimensional complex vector space. A holomorphic automorphic form with respect to a lattice $\Gamma \subset G$ is a holomorphic function $f : D \to V$ with the properties*

1) $f(\gamma z) = J_\sigma(\gamma, z) f(z)$,
2) *Some good behaviour at "infinity".*

We describe a canonical factor of automorphy in the Siegel case:

$$J_{\mathrm{can}}(M,Z) = CZ + D.$$

It is easy to check that this is an automorphy factor and we have

$$J_{\mathrm{can}}(M, \mathrm{i}E) = C\mathrm{i} + D \qquad (M \in \mathrm{O}(2n,\mathbb{R}) \cap \mathrm{Sp}(2n,\mathbb{R})).$$

Recall that $C\mathrm{i} + D$ has to be identified with M. Now have to consider a representation of $\mathrm{U}(n)$ on a finite dimensional complex vector space V. The universal property of the complexification show that they are in one-to-one correspondence to holomorphic representation $\sigma : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(V)$. The transformation formula of an automorphic form $f : \mathcal{H}_n \to V$ is

$$f(MZ) = \sigma(CZ + D)f(Z).$$

An important representation is the one dimensional $\sigma(A) = \det^r$ where r is an integer. The transformation formula in this case are

$$f(MZ) = \det(CZ + D)^r f(Z).$$

Chapter II. The Satake compactification

1. Minkowski reduction

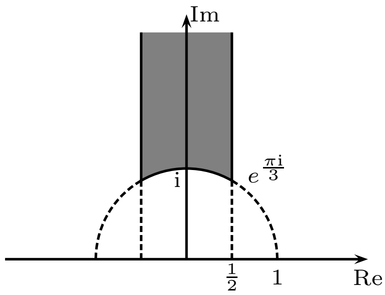
Our goal is to prove that $\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$. A maximal compact subgroup is $\mathrm{SO}(n,\mathbb{R})$. We start with the most classical case $n = 2$. We consider the upper half-plane $\mathcal{H}$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{H}$ from the left through Möbius transformations. We consider the natural map

$$\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathcal{H}, \quad z \longmapsto Mz = \frac{az + b}{cz + d}.$$

The stabilizer of i is $\mathrm{SO}(2,\mathbb{R})$. Hence this a maximal compact subgroup

$$\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) \xrightarrow{\ \sim\ } \mathcal{H}.$$

The space $\mathrm{SL}(2,\mathbb{R})\backslash\,\mathrm{SL}(2,\mathbb{R})$ has finite volume if and only if $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$ has finite volume with respect to the invariant measure. Obviously the measure $dxdy/y^2$ is invariant. So it is sufficient to construct a fundamental set of finite volume with respect to this measure.



The volume with respect to the invariant measure computes as

$$\int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty}\frac{dxdy}{y^2}=\int_{-1/2}^{1/2}\Big[-\frac{1}{y}\Big]_{\sqrt{1-x^2}}^{\infty}=\int_{-\sqrt{3}/2}^{\sqrt{3}/2}d\varphi=\pi\sqrt{3}.$$

The last integral transformation was $x=\sin(\varphi)$. Observe $dx=\cos(\varphi)d\varphi$ and $\sin(\sqrt{3}/2)=1/2$. The domain $\mathcal{F}$ is a fundamental domain. Therefore we get

$$\operatorname{vol}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H})=\pi\sqrt{3}.$$

There is a different model of $\mathrm{SL}(2,\mathbb{R})/\operatorname{SO}(2,\mathbb{R})$. It works for arbitrary n and gives a model for $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R})$. To define it, we introduce the space $\mathcal{P}_n$ of positive definite real symmetric $n\times n$-matrices. There is a natural action of $\mathrm{GL}(n,\mathbb{R})$ from the right

$$\mathcal{P}_n\times\mathrm{GL}(n,\mathbb{R})\longrightarrow\mathcal{P}_n,\quad (Y,A)\longmapsto Y[A]=A'YA.$$

We use this to define the map

$$\mathrm{GL}(n,\mathbb{R})\longrightarrow\mathcal{P}_n,\quad A\longmapsto AA'.$$

Two matrices A,B have the same image if and only if $A\mathrm{O}(n,\mathbb{R})=B\mathrm{O}(n,\mathbb{R})$. This means that we obtain a bijection

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R})\ \xrightarrow{\sim}\ \mathcal{P}_n.$$

Actually this map is topological. The group $\mathrm{GL}(n,\mathbb{Z})$ acts on $\mathrm{GL}(n,\mathbb{R})$ through multiplication from the left. The corresponding action is

$$\mathrm{GL}(n,\mathbb{Z})\times\mathcal{P}_n\longrightarrow\mathcal{P}_n,\quad (U,Y)\longmapsto Y[U']=UYU'.$$

Let

$$\mathcal{P}_n(1):=\{Y\in\mathcal{P};\quad \det Y=1\}.$$

A variant of the above action is

$$\mathrm{SL}(n,\mathbb{R})/\operatorname{SO}(n,\mathbb{R})\ \xrightarrow{\sim}\ \mathcal{P}_n(1).$$

In the case $n=2$ there must be a topological map $\mathcal{H}\to\mathcal{P}_2(1)$ such that the diagram

$$\begin{array}{ccc} & \mathrm{SL}(2,\mathbb{R})/\operatorname{SO}(2,\mathbb{R}) & \\ \swarrow & & \searrow \\ \mathcal{H} & \longrightarrow & \mathcal{P}_2(1) \end{array}$$

commutes. We compute the image of $z = x + \mathrm{i}y$. For this purpose we write

$$\begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} (\mathrm{i}).$$

the image of z computes as

$$Y = \begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} y + y^{-1}x & y^{-1}x \\ y^{-1}x & y^{-1} \end{pmatrix}.$$

The image of the fundamental domain $\mathcal{F}$ of the modular group $\mathrm{SL}(2, \mathbb{Z})$ computes as

$$2|y_1| \le y_2 \le y_0.$$

Since $\mathcal{P}_n$ is just the union of "surfaces" with fixed determinant we obtain that

$$\{Y \in \mathcal{P}_2; \quad 2|y_1| \le y_2 \le y_0\}$$

is a fundamental set of $\mathrm{SL}(2, \mathbb{Z})$ acting on $\mathcal{P}$ and from this follows that

$$\{Y \in \mathcal{P}_2; \quad 0 \le 2y_1 \le y_2 \le y_0\}$$

is a fundamental set of $\mathrm{GL}(2, \mathbb{Z})$ acting on $\mathcal{P}$. If one transforms this domain with $U \in \mathrm{GL}(2, \mathbb{Z})$ one obtains another fundamental set. Using the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we obtain the following result.

1.1 Lemma. *The set*

$$\{Y \in \mathcal{P}_2; \quad 0 \le 2y_1 \le y_1 \le y_2\}$$

is a fundamental set of $\mathrm{GL}(2, \mathbb{Z})$ *acting on* $\mathcal{P}_2$.

Minkowski's reduction theory provides a generalization to arbitrary n.

1.2 Definition. *A matrix $Y \in \mathcal{P}_n$ is called reduced in the sense of Minkowski if the following two conditions are satisfied.*

(M1) $Y[g] \ge y_k$ *for* $g \in \mathbb{Z}^n$ *and* $(g_k, \dots, g_n) = 1 \quad (1 \le k \le n)$.
(M2) $y_{k,k+1} \ge 0 \quad (1 \le k < n)$.

We denote the set of all Minkowski reduced matrices by $\mathcal{R}_n$.

1.3 Lemma. *Minkowski reduced matrices have the properties*

a) $0 < y_1 \le y_2 \le \dots y_n$,
b) $0 \le y_{\nu,\nu+1}$ *for* $1 \le \nu \le n$,
c) $2|y_{\mu\nu}| \le y_\nu$ *for* $\mu \ne \nu$.

We see that the domain $\mathcal{R}_2$ of Minkowski reduced matrices contains the fundamental set defined in Lemma 2.1.1. Hence $\mathcal{R}_n$ is a fundamental set at least in the case $n = 2$.

1.4 Remark. *Let Y be Minkowski reduced. Then*

a) $y_1 = m(Y)$.
b) *Let* $Y = \begin{pmatrix} Y_1 & * \\ * & * \end{pmatrix}$. *Then Y_1 is reduced too.*

1.5 Proposition. *The space $\mathcal{R}_n$ of Minkowski reduced matrices is a fundamental set of* $\mathrm{GL}(n, \mathbb{Z})$ *acting on $\mathcal{P}_n$.*

Proof. We start with an arbitrary $Y \in \mathcal{P}_n$. We will construct an equivalent matrix that satisfies (M1), (M2). It is suffient to obtain (M1), since then we can apply consecutively diagonal matrices which are obtained from the unit matrix by replacing one diagonal entry by -1. The conditions (M1)are obtained inductively, starting from $k = 1$ to $k = n$. Assume that we obtained already (M1) for $k < p$. We can apply a unimodular matrix of the form

Estimates of the space of Minkowski reduced matrices

We make use of the Jacobi decomposition of a matrix $Y \in \mathcal{P}_n$. Let $Y \in \mathcal{P}_n$ and let $n = p + q$. Then there exists a unique decomposition

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \begin{bmatrix} E_p & B \\ 0 & E_q \end{bmatrix}$$

with symmetric an positive $Y_1 \in \mathcal{P}_p$, $Y_2 \in \mathcal{P}_q$. Repeated application gives the Jacobi decomposition

$$Y = D[B], \quad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & b_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

So D is a diagonal matrix and B is a strictly upper triangular matrix. The formula

$$y_i = d_i + \sum_{\nu=1}^{i-1} d_\nu b_{\nu i}^2$$

shows that the Jacobi decomposition gives a topological map

$$\mathcal{P}_n \longrightarrow \mathbb{R}_{>0}^n \times \mathbb{R}^{n(n-1)/2}.$$

This formula implies also the *Hadamard inequality*

$$\det Y \le y_1 \cdots y_n.$$

1.6 Lemma (Gauss). *Every coprime vector $g \in \mathbb{Z}^n$ is the first column of a matrix in* $\mathrm{GL}(n, \mathbb{Z})$.

We omit the proof. There is another basic inequality for hermitian matrices. To formulate it, we introduce for $Y \in \mathcal{P}_n$

$$m(Y) = \min_{g \in \mathbb{Z}^n - \{0\}} Y[g].$$

It is clear that this minimum exists.

1.7 Proposition (Hermite). *For $Y \in \mathcal{P}_n$ the inequality*

$$m(Y) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} (\det Y)^{\frac{1}{n}}$$

holds.

Proof. We choose a $g \in \mathbb{Z}^n$ such that $Y[g] = m(Y)$. Since the vector g is coprime, it is the first column of a unimodular matrix U. We can replace Y by $Y[g]$. This means that we can assume

$$m(Y) = y_1.$$

We decompose

$$Y = \begin{pmatrix} y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \begin{bmatrix} 1 & b' \\ 0 & E_{n-1} \end{bmatrix}.$$

A simple calculation gives

$$Y[g] = y_1(g_1 + b'g_2)^2 + Y_2[g_2] \quad \text{for} \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \; g_1 \in \mathbb{Z}, \; g_2 \in \mathbb{Z}^{n-1}.$$

Now we choose g such that $m(Y_2) = Y_2[g_2]$, $|g_1 + b'g_2| \leq 1/2$. Then we get

$$y_1 = m(Y) \leq Y[g] \leq \frac{1}{4}y_1 + m(Y_2),$$

or

$$y_1 \leq \frac{4}{3}m(Y_2).$$

The claim now follows from inudction. □

The basic estimate for the space of Minkowski reduced matrices is as follows.

1.8 Theorem. *There exists a constant C_n such that*

$$\det Y \leq y_1 \cdots y_n \leq C_n \det Y$$

for all Minkowski reduced matrices.

Proof. Induction by n. We assume that $C_1 \ldots, C_{n-1}$ have been constructed. We prove the existence of C_n indirectly and assume. That there exists a sequence of matrices $Y = Y(\nu) \in \mathcal{R}_n$ such that

$$\frac{y_1 \cdots y_n}{\det Y} \longrightarrow \infty \qquad (\nu \longrightarrow \infty).$$

From the Hermite inequality we can deduce that not all y_{k+1}/y_k are bounded. We choose k maximal that this sequence is unbounded. We may assume that $y_k/y_{k+1} \to 0$ for $\nu \to \infty$. Using this k we decompose

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \begin{bmatrix} E_k & B \\ 0 & E_{n-k} \end{bmatrix}$$

to obtain

$$Y[g] = Y_1[g_1 + Bg_2] + Y_2[g_2] \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

We choose g such that $Y_2[g_2] = m(Y_2)$ and such that the components of g_1+Bg_2 are in $[-1/2, 1/2]$. Since Y_1 is reduced too, we get

$$y_{k+1} \leq \frac{k^2}{4} y_k + m(Y_2).$$

The Hermite inequality for Y_2 and the fact that y_k/y_{k+1} tends to 0 gives

$$y_{k+1} \leq C(\det Y_2)^{1/n-k}$$

for suitable C.The induction hypothesis says

$$y_1 \cdots y_k \leq C_k \det Y_1.$$

From the choice of K we know that that $y_{k+1} \cdots y_n$ is bound by y_{k+1}^{n-k} up to a constant factor. Hence there exist constants C', C with the property

$$y_1 \cdots y_n \leq C'(\det Y_1) y_{k+1}^{n-k} \leq C \det Y_1 \det Y_2 = C \det Y.$$

This is a contradiction. □

1.9 Definition. *For a positive number u let $\mathcal{R}_n[u]$ be the set of all positive matrices Y with the properties*

a) $y_\nu < u y_{\nu+1}, \quad 1 \neq \nu < n,$
b) $|y_{\mu\nu}| < u y_\nu, \quad 1 \leq \mu, \nu \leq n,$
c) $y_1 \cdots y_n < u \det Y.$

The domains $\mathcal{R}(u)$ are open and one has

$$\mathcal{R}_n[u] \subset \mathcal{R}_n[u'] \text{ for } u < u' \quad \text{and} \quad \bigcup_{u>0} \mathcal{R}_n[u] = \mathcal{P}_n.$$

In particular, every compact subset of $\mathcal{P}_n$ is contained in $\mathcal{R}_n[u]$ for big enough u.

1.10 Proposition. *For big enough u the domain $\mathcal{R}_n[u]$ contains the space of Minkowski reduced matrices.*

We need a variant of this definition.

1.11 Definition. *For a positive number u, let be $\mathcal{R}_n(u)$ be the domain of all positive matrices*

$$Y = D[B], \quad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & b_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

with the properties

a) $d_1 < ud_2, \dots, d_{n-1} < ud_n$,
b) $|b_{ij}| < u$ *for* $1 \le i \le j \le n$.

The domains $\mathcal{R}_n(u)$ are open and one has

$$\mathcal{R}_n(u) \subset \mathcal{R}_n(u') \text{ for } u < u' \quad \text{and} \quad \bigcup_{u>0} \mathcal{R}_n(u) = \mathcal{P}_n.$$

1.12 Lemma. *There exists a constant $\delta = \delta(n,u)$ with the property $\delta D < Y < \delta^{-1} D$.*

Proof. Consider the matrix $D^{-1/2} Y D^{-1/2}$. It is bounded and its determinant is one. This implies that its eigenvalues are bounded from above and below by positive constants. This shows that there exists a constant δ with the property $\delta D < Y < \delta^{-1} D$. □

1.13 Lemma. *For every positive number u there exists $\tilde{u}$ such that*

$$\mathcal{R}_n(u) \subset \mathcal{R}_n[\tilde{u}], \quad \mathcal{R}_n[u] \subset \mathcal{R}_n(\tilde{u}).$$

Proof. 1) Let $Y \in \mathcal{R}_n(u)$. Conditions a),b) in 1.9 are easy. Condition c) follows from Lemma 1.12.

2) We conclude by induction. So we assume that the existence of $\tilde{u}$ for $n-1$ instead of n has been verified. We have to make use of

$$y_{\mu\nu} = \sum_{i=1}^{n} d_i b_{i\mu} b_{i\nu}.$$

From $y_1 \dots y_n \le u \det Y = d_1 \dots d_n$ and from the induction hypothesis it is easy to deduce condition a) in Definition 1.11. It remains to prove that $b_{\nu n}$ is bounded. This follows inductively from $\sum_{i=1}^{n} d_i b_{i\nu} b_{in} = y_{in}$. □

As a preliminary stage of a compactification we introduce a notion of convergence. Later we will see that it is related to a topology.

1.14 Definition. *A sequence (Y_m) in $\mathcal{P}_n$ tends to ∞ if the following two conditions hold.*

a) *There exists $u > 0$ such that all Y_m are contained in $\mathcal{R}_n(u)$.*
b) *The smallest eigen value of Y_m tends to ∞.*

Instead of b) one can demand
b') $d_1 = y_1 \to \infty$.

1.15 Lemma. *Let (Y_m), $(\tilde{Y}_m)$ be two sequences in $\mathcal{P}_n$ such that the sequence $\tilde{Y}_m - Y_m$ is bounded. If one of them converges to ∞ then both do.*

Proof. We have to show that there exists an u such that all $\tilde{Y}_m$ are contained in $\mathcal{R}_n$. All what we need is an estimate of the form

$$\det Y \leq C \det \tilde{Y}.$$

(We omit the index m in the notation.) We use the Jacobi decomposition $Y = D[B]$. Set $W = (\tilde{Y} - Y)[B^{-1}]$. We have to show that $\det B$ is bounded up to a constant factor by $\det(D + W)$. In other words, $\det(E + D^{-1}W)$ is bounded from below by a positive constant. This is clear since $D^{-1}B$ tends to zero. □

Definition 1.14 is a special case of the following Definition ($m = 0$).

1.16 Definition. *A sequence*

$$Y(\nu) = \begin{pmatrix} Y_1(\nu) & 0 \\ 0 & Y_2(\nu) \end{pmatrix} \begin{bmatrix} E_m & B(\nu) \\ 0 & E_{n-m} \end{bmatrix}$$

converges to $Y_1 \in \mathcal{P}_m$ if the following conditions are satisfied.
1) *The sequence $B(\nu)$ is bounded.*
2) *We have $Y_1(\nu) \to Y_1$ in the usual sense.*
3) *$Y_2(\nu) \to \infty$.*

1.17 Lemma. *If a sequence is convergent in the sense of Definition 1.16, then it is contained in $\mathcal{R}_n(u)$ for suitable u.*

Proof. We have to show that the sequence is contained in a suitable $\mathcal{R}_n(u)$. For this one uses the Jacobi decomposition

$$Y = D[W], \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad W = \begin{pmatrix} W_0 & W_1 \\ 0 & W_2 \end{pmatrix},$$

explicitely

$$Y_1 = D_1[W_0], \quad Y_1 B = W_0' D_1 W_1, \quad Y_1[B] + Y_2 = D_1[W_1] + D_2[W_2].$$

We obtain the convergence of Y_1, hence of $D_1, D_1^{-1}, W_0, W_0^{-1}$ and hence the boundedness of W_0. From the boundedness of $Y_1 B$ we get the boundedness of W_1. The convergences $D_2[W_2] \to \infty$ implies the boundedness of W_2. The rest is clear. □

1.18 Proposition. *Let $Y(\nu)$ be a sequence in some $\mathcal{R}_n(u)$ and such that $d_1(\nu) = y_1(\nu)$ is bounded from below by a positive constant. Then there exists a sub-sequence that converges to some $Y_1 \in \mathcal{P}_m$ for suitable m. (We set $\mathcal{P}_0 = \{\infty\}$).*

Proof. Just take for m the biggest m such that $d_1, \dots, d_m$ are bounded. □

1.19 Theorem. *For every $u > 0$ the domain $\mathcal{R}_n(u)$ (similarly $\mathcal{R}_n[u]$) has the finiteness property. The set of all $U \subset \mathrm{GL}(n, \mathbb{Z})$ with the property*

$$\mathcal{R}_n(u) \cap \mathcal{R}_n(u)[U] \neq \emptyset$$

is finite.

2. The Siegel fundamental domain

2.1 Definition. *The Siegel fundamental domain $\mathcal{F}_n$ consists of all $Z \in \mathcal{H}_n$ such that the following conditions are satisfied.*

(S1) $|\det(CZ + D)| \geq 1$ *for all* $M \in \Gamma_n$.
(S2) Y *is reduced in the sense of Minkowski.*
(S3) X *is reduced mod 1, i.e.* $2|x_{\mu\nu}| \leq 1/2$ $(1 \leq \mu, \nu \leq n)$.

Siegel proved that $\mathcal{F}_n$ is a fundamental domain. For our purposes the following result is sufficient.

2.2 Proposition. *The domain $\mathcal{F}_n$ is a fundamental set for Γ_n.*

Similar to the Minkowski domain we will make use of an open estimate of $\mathcal{F}_n$.

2.3 Definition. *For positive $u > 0$ the domain $\mathcal{F}_n(u)$ is defined through*

1) $|x_{\mu\nu}| \leq u$ *for* $1 \leq \mu, \nu \leq n$,
2) $Y \in \mathcal{R}_n(u)$,
3) $1 < uy_1$.

2.4 Proposition. *The domains $\mathcal{F}_n(u)$ are open. Their union is the full half plane $\mathcal{H}_n$. For big enough u they contain the Siegel fundamental domain. For each u there are only finitely many $M \in \Gamma_n$ such that*

$$M(\mathcal{F}_n(u)) \cap \mathcal{F}_n(u) \neq \emptyset.$$

3. The Satake compactification as topological space

The idea of the Satake compactification is to define

$$\lim_{t\to\infty}\begin{pmatrix} Z & 0 \\ 0 & \mathrm{i}tE \end{pmatrix} = Z.$$

Having this in mind we introduce the disjoint union

$$\overline{\mathcal{H}_n/\Gamma_n} = \mathcal{H}_n/\Gamma_n \cup \ldots \cup \mathcal{H}_0/\Gamma_0.$$

Here we understand $\mathcal{H}_0/\Gamma_0$ to be a single point ∞.

3.1 Theorem (Satake). *On $\overline{\mathcal{H}_n/\Gamma_n}$ there is a unique structure as Hausdorff space with countable basis of the topology such that the following condition holds. A sequence $a(\nu)$ in $\mathcal{H}_n/\Gamma_n$, $m \le n$, converges to a point a in $\mathcal{H}_j/\Gamma_j$ if and only if $j \le m$ and if there exists a $u > 0$ and representants $Z(\nu) \in \mathcal{F}_m(u)$ of a_ν and a representant Z_1 of $a \in \mathcal{H}_j$ such that*

$$Z(\nu)^{-1} \longrightarrow \begin{pmatrix} Z_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

in the usual sense. This space is compact.

4. The notion of a complex space in the sense of Serre

We introduce the notion of a concrete ringed space.

4.1 Definition. *A concrete ringed space $(X, \mathcal{O}_X)$ is a topological space together with a subsheaf of $\mathbb{C}$-algebras $\mathcal{O}_X \subset \mathcal{C}_X$.*

Here $\mathcal{C}_X$ denotes the sheaf of complex valued continuous functions. Subsheaf means that for each open subset $U \subset X$ a subring $\mathcal{O}_X(U) \subset \mathcal{C}_X(U)$ is given. We assume the following properties.

1) The constant functions are in $\mathcal{O}_X(U)$.
2) Let $V \subset U$ be open sets. Then for $f \in \mathcal{C}_X(U)$ the restriction $f|V$ is contained in $\mathcal{O}_X(V)$.
3) Assume that $U = \bigcup U_i$ is an open covering of an open subset of X. Let f be a function such that for all i its restrictions to U_i are in $\mathcal{O}_X(U_i)$ then f is in $\mathcal{O}_X(U)$.

4.2 Definition. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of concrete ringed spaces is a continuous map with the property. For all open subsets $V \subset Y$ and all $h \in \mathcal{O}_Y(V)$ one has $h \circ f \in \mathcal{O}_X(f^{-1}(U))$.*

It is clear that the identity map defines a morphism $\mathrm{id} : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$. The composition of two morphisms is a morphism. Hence the notion of isomorphism of concrete ringed spaces is explained. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if it is topological and if it induces for any open subset $U \subset X$ a bijection between $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ where $V = f(V)$.

Let Y be a subset of a concrete ringed space $(X, \mathcal{O}_X)$. We equip Y with a structure $\mathcal{O}_Y$ of a concrete ringed space. The topology of Y is the induced topology. Let $V \subset Y$ an open subset. We define:

A function $f : V \to \mathbb{C}$ on some open subset of X belongs to $\mathcal{O}_Y(V)$ if for every $a \in W$ there exists an open neighborhood $a \in \tilde{W} \subset X$ and a function $h \in \mathcal{O}_X(\tilde{W})$ such that $f(x) = h(x)$ for all $x \in U \cap W$.

It is clear that that $(Y, \mathcal{O}_Y)$ is a concrete ringed space. Such a space is called a *subspace* of $(X, \mathcal{O}_X)$. In the case that Y is an open subset of X the definition can be made easier. In this case on has $\mathcal{O}_Y(V) = \mathcal{O}_X(V)$.

The canonical injection $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism and moreover the following is true:

4.3 Remark. *Let $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ be concrete ringed spaces and $(Y, \mathcal{O}_Y)$ a concrete ringed subspace of $(X, \mathcal{O}_X)$. Let $f : Z \to Y$ be a continuous map. Then $f : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ is a morphism if and only if the composition with the canonical injection is a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$.*

We mention without proof that the notion of subspace is transitive: If $(Z, \mathcal{O}_Z)$ is a concrete ringed subspace of $(Y, \mathcal{O}_Y)$ and $(Y, \mathcal{O}_Y)$ is a concrete ringed subspace of $(X, \mathcal{O}_X)$ then $(Z, \mathcal{O}_Z)$ is a concrete ringed subspace of $(X, \mathcal{O}_X)$.

The following simple remark indicates that the notion of morphism of a concrete ringed space is useful:

4.4 Remark. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subset. We equip them with the sheaves $\mathcal{O}_U, \mathcal{O}_V$ of holomorphic functions in the usual sense. A map $f : U \to V$ is holomorphic in the usual sense if and only if it as morphism $(U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$.*

In the following we understand by $\mathcal{O}_{\mathbb{C}^n}$ always the sheaf of holomorphic function in the usual sense. Let $U \subset \mathbb{C}^n$ be an open set and let $f_1, \dots, f_m$ be holomorphic functions on U. Set

$$A = \{z \in U; \quad f_1(z) = \cdots = f_m(z) = 0.$$

We equip A with the restricted structure $\mathcal{O}_A$. We call $(A, \mathcal{O}_A)$ a *model space*. Notice that $(U, \mathcal{O}_U)$ is a model space.

4.5 Definition. *A complex space $(X, \mathcal{O}_X)$ in the sense of Serre is a concrete ringed space such that for each point the exists an open neighborhood U and and a model space $(A, \mathcal{O}_A)$ such that the concrete ringed subspace $(U, \mathcal{O}_U)$ and $(A, \mathcal{O}_A)$ are isomorphic concrete ringed spaces.*

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces. A map $f : X \to Y$ is called a holomorphic map if it is a morphism of concrete ringed spaces. Clearly $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a complex space. For a function $f : X \to \mathbb{C}$ the following two conditions are equivalent:

1) $f \in \mathcal{O}_X(X)$,
2) $f : (X, \mathcal{O}_X) \to (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is a morphism.

The simple proof is left to the reader. Since we use the notation $\mathcal{O}_X$ for the structure sheaf of a complex space, there is no need to mention it always. Hence we frequently write X instead of $(X, \mathcal{O}_X)$.

Another simple property of complex spaces is the following. Let $(X, \mathcal{O}_X)$ be complex space, let $U \subset X$ be an open subset and $f \in \mathcal{O}_X(U)$ a holomorphic function without zeros. Then $1/f \in \mathcal{O}_X(U)$.

The notation of analytic subsets generalizes to complex spaces: A subset $Y \subset X$ of a complex space is called analytic if the following condition is satisfied: For each $a \in Y$ there exists an open neighborhood $a \in W \subset X$ and finitely many $f_1, \dots, f_n \in \mathcal{O}_X(W)$ such that

$$Y \cap W = \{x \in W; \quad f_1(x) = \cdots = f_n(x) = 0\}.$$

Open subsets are very special cases of analytic subsets. (Take $f_i = 0$.)

4.6 Lemma. *If Y is an analytic subset of an complex space. Then one easily can show that the subspace $(Y, \mathcal{O}_Y)$ is a complex space as well.*

Such a space is called a complex subspace of X.

Usually one only considers only complex subspaces that either are open or closed. This is sufficient because of the following simple

4.7 Remark. *Let $Y \subset X$ be an analytic subset of the complex space X. There exists an open subset $U \subset X$ that contains Y and such that Y is closed in U.*

4.8 Lemma. *Let A, B be two closed analytic subsets of a complex space. Then $A \cap B$ and $A \cup B$ are analytic too.*

4.9 Lemma. *Let $f : X \to Y$ be a holomorphic map of complex spaces and $B \subset Y$ a closed analytic subset. Then $f^{-1}(B)$ is analytic too.*

Regular locus

A point a of a complex space $(X, \mathcal{O}_X)$ is called smooth if there exists an open neighborhood U and an open subset $V \subset \mathbb{C}^n$ such that $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_{\mathbb{C}^n}|V)$ are isomorphic. An analytic manifold is a complex space where all points are smooth. Let $(X, \mathcal{O}_X)$ be a complex space and let X_0 be the set of all smooth points. We denote by $S = X - X_0$ the set of singular (=non smooth) points and call it the singular locus of X. A basic result of local complex analysis states the following.

4.10 Theorem. *The regular locus X_0 of a complex space is open and dense. The singular locus S is a closed analytic subset.*

Irreducibility

4.11 Definition. *A complex space X is called **irreducible** if it is not possible to write X as union of two closed analytic subsets different from X.*

4.12 Propsotion. *A complex space is irreducible if and only of its regular locus is connected.*

4.13 Definition. *An irreducible component of a complex space X is a maximal closed analytic irreducible subspace.*

4.14 Proposition. *Every complex space X is the union of its irreducible components. For every compact subset K of X there exist only finitely many irreducible components that intersect K.*

Example The zero locus of $z_1z_2 = 0$ in $\mathbb{C}^2$ is a complex space. It has two irreducible components $z_i = 0$.

5. Dimension of complex sapces

We want to define the dimension $\dim_a X$ of a complex space X at a point a. There is no problem if X is an analytic manifold. The dimension $\dim_a X$ is locally constant on an analytic manifold, hence constant if X is a connected analytic manifold. For a not necessarily connected analytic manifold we define the dimension to be the supremum of the dimensions of all connected components (or of all $\dim_a X$ if one prefers this). This dimension can be infinite. Now let X be an arbitrary complex space and $a \in X$. Then there exists an open neighborhood U such that its regular locus U_0 has finite dimension. We do not give details. We just mention that this follows from the map that one has an holomorphic injection of some open neighborhood into some $\mathbb{C}^n$. So we can define

$$\dim_a X = \min_U \dim(U_0) < \infty$$

where U runs through all open neighborhoods of a. We define

$$\dim X = \sup_a \dim_a X.$$

We call X pure dimensional if $\dim_a X$ is constant on X. For example irreducible complex spaces are pure dimensional.

5.1 Proposition. *Let $f : X \to Y$ be a surjective holomorphic map between complex spaces. Then* $\dim Y \leq \dim X$.

5.2 Proposition. *Let $f : X \to Y$ be a surjective holomorphic map between complex spaces. For all points $x \in X$ and $y = f(x)$ we have*

$$\dim_x X \leq \dim_y Y + \dim f^{-1}(y).$$

6. Extension theorems

Let $(X, \mathcal{O}_X)$ be a ringed space. For any point one can define the "ring of germs"

$$\mathcal{O}_{X,x} = \varinjlim \mathcal{O}_X(U)$$

where U runs through all open neighborhoods of a. Its elements are equivalence classes of pairs (U, f) where U is an open neighborhood do a and $f \in \mathcal{O}_X(U)$. Two such (U_1, f_1), (U_2, f_2) are called equivalent if there is an open neighborhood $a \subset V \subset U_1 \cap U_2$ such that $f_| V = f_2|V$. It is clear how to define a structure as $\mathbb{C}$-algebra.

A ring R is called normal if it is an integral domain and if it is integral closed in its field of fractions K. This means that $x \in K, \quad x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0, \quad r_i \in R$ implies $x \in R$. Examples of normal rings are $\mathbb{Z}$, polynomial rings in arbitrary many variables over a field, the ring of convergent power series. A ringed space is called normal if all rings $\mathcal{O}_{X,a}$ are normal. In the following theorem we agree that the dimension od the empty complex space is $-\infty$.

6.1 Theorem. *Let S be the singular locus of a normal complex space X. Then*

$$\dim_s S \le \dim_s X - 2$$

for all $s \in S$. In particular, a normal complex space of pure dimension one is smooth (a Riemann surface).

Riemann extension theorem for normal complex spaces

6.2 Theorem. *Let X be a normal complex space and let $A \subset X$ be a thin closed analytic subset of X. Let $f : X \to \mathbb{C}$ be a continuous function such that $f|(X - S)$ is analytic. Then f is analytic.*

Analytic manifolds are normal. This shows a more classical form of the Riemann extension theorem. The following theorem is an improvement of the Riemann extension theorem in the case that the codimension of A is big enough.

6.3 Theorem. *Let X be a normal complex space and let $A \subset X$ be a closed analytic subset with the property*

$$\dim_a A \le \dim_a X - 2 \quad \textit{for all } a \in A.$$

Then every holomorphic function on $X - A$ extends to a holomorphic function on X.

Let X be a complex space and $U \subset X$ an open dens subset. A holomorphic function f on U is called meromorphic on X if for every point $a \in X - U$ there exists an open neighborhood $U(a)$ and holomorphic functions φ, ψ on $U(a)$ with the property $\psi(x) \ne 0$ for $x \in U(a) \cap U$ and such that $f(x) = \varphi(x)/\psi(x)$ for $x \in U(a) \cap U$. There exists a biggest open domain $D(f)$, $U \subset D(f) \subset X$ such that f extends holomorphically to $D(f)$. It can be shown that the complement of $D(f)$ is a closed analytic subset.

We consider two meromorphic function (U, f), (V, g) to be equal if their maximal extensions agree, or, equivalently, if they agree on $U \cap V$. (Actually one should consider equivalence classes.) The set of all meromorphic functions $K(X)$ is a ring and even a field if X is irreducible.

Since open subsets of complex spaces are complex spaces too, we can talk about meromorphic functions on open subsets of a complex space.

6.4 Theorem. *Let X be a normal complex space and let $A \subset X$ be a closed analytic subset with the property*

$$\dim_a A \le \dim_a X - 2 \quad \textit{for all } a \in A.$$

Then every meromorphic function on $X - A$ extends to a meromorphic function on X.

6.5 Theorem (Remmert, Stein). *Let X be a complex space $S \subset X$ a closed analytic subset and $A \subset X - S$ a closed analytic subset. Assume*

$$\dim_a A > \dim X \quad \textit{for all } a \in A.$$

Then the closure $\bar{A}$ of A in X is an analytic subset of X. One has $\dim \bar{A} = \dim A$.

A continuous map $f : X \to Y$ between locally compact spaces is called *proper* if the inverse image of a compact set is compact. It is called *finite* if it is proper and if all fibres are finite sets.

6.6 Theorem (Remmert). *Let $f : X \to Y$ be a holomorphic and proper map between complex spaces. Then the image is a closed analytic subset of Y.*

Normalization

Let X be a complex space. A normalization of X is a pair $(\tilde{X}, q)$ consisting of a normal complex space and a finite and surjective holomorphic map $f : \tilde{X} \to X$ such there exists a thin closed analytic subset $A \subset X$ such that the inverse image $\tilde{A} \subset \tilde{X}$ is thin as well and such that $\tilde{X} - \tilde{A} \to X - A$ is biholomorphic.

6.7 Theorem. *Every complex space X admits a normalization $q : \tilde{X} \to X$. It is essentially unique and has the following universal property. Let Y be a normal complex space and $f :\to X$ a holomorphic map. Assume that for any open subset $U \subset X$ and any thin closed subset $A \subset U$ the inverse image $f^{-1}(A)$ is thin in $f^{-1}(U)$. Then there exists a holomorphic map $\tilde{f} : Y \to \tilde{X}$ that lifts f.*

7. The analytifcation of the Satake compactification

The basic extension criterion is due to Baily.

7.1 Theorem (Baily). *Let $(X, \mathcal{O}_X)$ be a concrete ringed space and let $X_0 \subset X$ be an open dense subspace. We assume that the following conditions are satisfied.*

1) *$(X_0, \mathcal{O}_X|X_0)$ is a normal complex space of finite dimension. For open $U \subset X$ the ring $\mathcal{O}_X(U)$ consists of all continuous functions whose restriction to $U_0 = X_0 \cap U$ are contained in $\mathcal{O}_X(U_0)$.*
2) *Every point $a \in X$ admits an open neighborhood U such that the functions in $\mathcal{O}_X(U)$ separate the points in $U_0 = X_0 \cap U$.*
3) *Every point $a \in X$ admits a fundamental system of open neighborhoods U such that $U_0 = X_0 \cap U$ is connected.*

4) *The ringed subspace* $(S, \mathcal{O}_S)$ *where* $S = X - X_0$ *is a complex space.*
5) $\dim S < \dim_a X_0$ *for all* $a \in X_0$.

Then $(X, \mathcal{O}_X)$ *is a normal complex space and* S *is an analytic subset.*

8. Algebraization of the Satake compactification

The projective space $P^n\mathbb{C}$ consists of all one-dimensional subspaces of $\mathbb{C}^{n+1}$. There is a surjective map $\mathbb{C}^{n+1} - \{0\} \to P^n(\mathbb{C})$. We equip $P^n\mathbb{C}$ with the quotient topology and the following structure $\mathcal{O}_{P^n\mathbb{C}}$ as ringed space. A function on an open subset $U \subset P^n\mathbb{C}$ is called holomorphic if its pull back to the inverse image $\tilde{U} \subset \mathbb{C}^{n+1} - \{0\}$ is holomorphic in the usual sense. It is easy to show that $P^n\mathbb{C}$ is a compact connected complex manifold of dimension n.

Let $P \in \mathbb{C}[z_0, \dots, z_n]$ be a homogenous polynomial. Then it makes sense to talk about the zero set of P in $P^n\mathbb{C}$. A subset $X \subset P^n\mathbb{C}$ is called *projective algebraic* if it is the set of joint zeros of a set of homogenous polynomials. By Hilbert's basis theorem it is always possible to restrict to finite sets of polynomials. Clearly a projective algebraic set is a complex space.

8.1 Theorem (Chow). *Every closed analytic subset of* $P^n\mathbb{C}$ *is projective algebraic.*

Combining this with the mapping theorem of Remmert, we obtain the following result.

8.2 Theorem. *Let* $f : X \to P^n\mathbb{C}$ *be a holomorphic map of a compact complex space into the projective space, then its image is projective algebraic.*

The Zariski topology of $P^n\mathbb{C}$ is defined such that the projective algebraic subsets are the closed subsets. A subset X of $P^n\mathbb{C}$ is called *quasi-projective* if it is locally closed with respect to the Zariski topology. This means that the X can be written in the form $Y - A$ where Y is a projective algebraic subset and $A \subset Y$ is a projective algebraic set too.

The category of quasi-projective varieties

Let $X \subset P^n\mathbb{C}$ and $Y \subset P^m\mathbb{C}$ be projective varieties. A map $f : X \to Y$ is called *regular* if for every point $a \in X$ there exists a Zariski open set $U \subset P^n\mathbb{C}$ and $m + 1$ homogenous polynomials $P_0(z_0, \dots, z_n), \dots P_m(z_0, \dots, z_n)$ which have no joint zero on U and such that

$$f(x) = (P_0(x), \dots P_m(x)) \quad \text{for } x \in X \cap U.$$

Quasiprojective sets are complex spaces and regular maps are holomorphic.

Graded algebras

Let A be a $\mathbb{C}$-algebra (commutative, assoziative and with unit). A graduation of A is a sequence of sub vector spaces $A_0, A_1, \ldots$ such that

$$A = \bigoplus_{n=0}^{\infty} A_n.$$