RECIPROCITY LAWS. FROM EULER TO EISENSTEIN

REVIEWER P. SHIU

Fermat found that primes $p \equiv 1 \pmod 4$ are sums of two squares, and Euler went on to investigate the representation of primes using more general quadratic forms. This led to the problem of determining whether a given prime p is a square modulo another prime q. After much effort by Euler and Legendre the law of quadratic reciprocity was formulated, relating the answer to whether q is a square modulo p. The first proof of the law was given by Gauss in his *Disquisitiones Arithmeticae* (1801); he called the law the "queen of number theory"[1] and gave seven more proofs later in his life. Most elementary textbooks on number theory contain 'Gauss's Lemma', which was used in his third and fifth proofs, and is the basis of many 'simple' proofs. Students do find the application of the law to determine the value of a given Legendre symbol nothing short of miraculous, but it has to be said that most of them also find that the proof of the law, however simple, does not really tell them why it is true. Actually, in order to have some real understanding of the law, one has to know a fair amount of algebraic number theory and to learn some deep mathematics, and this is what the book being reviewed is all about. From the preface, the author makes it clear that the law should be understood in terms of algebraic number theory. After a short chapter on the genesis of quadratic reciprocity, he moves quickly on to quadratic and cyclotomic number fields. Much of the book is concerned with power residues and the associated 'higher reciprocity law', which were a central theme in nineteenth century number theory. In the hands of twentieth century mathematicians, the subject matter has been developed into 'class field theory', and the author intends to write a second volume, no doubt giving the readers the finer points on the Artin reciprocity law.

This wonderful book is an excellent expository account of some difficult and deep mathematics, and is the beautiful work of a fine scholar. Each chapter contains long notes, mostly for historic references but often also containing much information on many related topics; there are also exercises and references which supplement the 885 items listed at the end of the book. There are three appendices, the first one giving the dates of the people involved from Pierre de Fermat to Emma Lehmer. The second appendix gives the chronology of 196 proofs, from the incomplete one by Legendre in 1788 to the one by the author in 2000, and the third appendix contains some open problems. Incidentally, when reading the article by M. Gerstenhaber with the intriguing title 1The 152nd proof of the law of quadratic reciprocity' many years ago in the *American Mathematical Monthly* (**70** (1963), pp. 397–398), I had wondered how he knew it was the '152nd'. The answer is given in the first page of the preface, and the author has listed this proof as his 149th entry.

[1] Actually he called it 'theorema aureum', the golden theorem. He once called number theory the queen of mathematics. [FL]