On the Arf invariant in historical perspective, Part 2.

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1 Introduction

This is a continuation of our previous article [LR10] in this journal, where we discussed the paper (1941) of Cahit Arf [Arf41]. There he introduced what today is called the "Arf invariant" of a quadratic form over a field of characteristic 2.

After our article had appeared we obtained some new information about the present state of the theory. When we say "new" then this means that this was new to us, i.e., we had not been aware before of the literature on the subject. (See our list of references.) It seems worthwhile to us to report here about it, as far as it is relevant for the assessment of Arf's paper in its historical perspective.[1]

2 The situation

Let us first explain the situation. We assume that the reader is aware of our previous article [LR10]. In the following this will be cited as "Part 1".

K always denotes a field of characteristic 2 if nothing is said to the contrary.

The aim of Arf's paper of 1941 was to study the relations between the theory of quadratic forms over K and the theory of central simple algebras over K. In characteristic $\neq 2$ this had been done in Witt's seminal paper of 1937 [Wit37], and Arf wished to treat the analogue in characteristic 2. His Arf invariant of a quadratic form is the analogue in characteristic 2 of the discriminant in characteristic $\neq 2$.

Arf showed that in characteristic 2 the *Clifford algebra* of a quadratic form plays a similar role to the *Hasse algebra* which had been introduced by Witt in characteristic $\neq 2$. The Clifford algebra is a product of quaternion algebras, hence it is of exponent 2 in the Brauer group of the base field K.

Arf stated a condition for the Brauer group of K which was to imply that every nonsingular[2] quadratic form is uniquely determined by its three invariants: **Dimension**, **Arf invariant**, and **Clifford algebra**. Arf's condition reads as follows:

(Q): *The quaternion algebras over K form a subgroup of $Br(K)$.*

[1] We are indebted to Detlev Hoffmann for providing us with this information. Also we would like to thank K. J. Becher, K. Conrad and S. Garibaldi for their help in this matter.

[2] If q is a quadratic form over K and V the corresponding quadratic space then q (or V) is called "nonsingular" if $V^{\perp} = 0$, i.e., there is no nonzero element in V which is perpendicular to every element in V. Arf used in his paper [Arf41] the terminology "*vollregulär*" (in German) which we had translated in Part 1 as "completely regular". But here we prefer to use "nonsingular" which seems to be the established terminology. See, e.g., [Bae82].

In other words: *If A and B are quaternion algebras over K then $A \otimes B \sim C$ where C is again a quaternion algebra.*[3]

Arf divided his main result into two theorems.

Arf's first theorem. *If K satisfies condition* (Q) *then every nonsingular quadratic form over K of dimension > 4 is isotropic.*

Today this can be formulated in terms of the so-called u-invariant $u(K)$ which is defined as the maximal dimension of nonsingular anisotropic quadratic forms over K. Thus Arf's first theorem claims that condition (Q) implies $u(K) \le 4$.

Arf's second theorem. *If $u(K) \le 4$ then every nonsingular quadratic form over K is determined* (*up to isomorphism*) *by its three invariants: Dimension, Arf invariant, and Clifford algebra.*

Now, as pointed out in Part 1 there is an error in Arf's proof of his first theorem.[4] We presented a counterexample showing that his proof does not work. On the other hand we formulated a stronger condition which permits one to prove his first theorem, namely:

(E): *The field K has imperfectness degree $e(K) \le 1$.*

The imperfectness degree e of a field K of characteristic 2 is defined by the formula $[K : K^2] = 2^e$. Hence $e(K) \le 1$ means that either K is perfect ($e = 0$) – in which case there is only one quaternion algebra over K, namely the splitting one, and then it turns out that every nonsingular quadratic space over K of dimension > 2 is isotropic – or else, in case $e(K) = 1$ there is only one inseparable quadratic extension of K, namely $K^{1/2}$ – which then is a splitting field of every quaternion algebra. In this case too it is not difficult to prove Arf's first theorem, as we have shown in Part 1.

Thus our correction of Arf's result consists in replacing condition (Q) by the stronger condition (E):

Our correction of Arf's first theorem. *If K satisfies condition* (E) *then every nonsingular quadratic form over K of dimension > 4 is isotropic.* In other words: $e(K) \le 1$ implies $u(K) \le 4$.

[3] The tensor product is to be understood over K. The sign $\sim$ indicates Brauer equivalence of central simple algebras over K.

[4] This error has been detected by Baeza, as indicated in [AJ95].

Arf had written his paper at the suggestion of Hasse. The latter was mainly interested in the case when K is an algebraic function field of one variable with perfect field of constants k. For those fields the validity of condition (E) was well known at the time. In a letter to Hasse dated March 29, 1940, Arf answered a question of Hasse who wished to know whether an algebraic function field of one variable with a perfect field of constants satisfies (Q). In that letter Arf showed that, indeed, (E) implies (Q).

All the details of our (easy) proof of Theorem 1a in Part 1 were known to Arf. In fact they were contained in his paper. So why did he not use (E), instead of the weaker condition (Q) which complicated the proof considerably, so much that he did not detect his error? (And Hasse too did not find it.)

Of course we will never know. One explanation would be that he was not aware of the different behavior of quaternion algebras with respect to their separable and to their inseparable quadratic splitting fields. This was discovered much later only. Let us explain:

3 Separable and inseparable splitting fields

Let A, B be quaternion algebras over K. If A, B have a common quadratic splitting field then $A \otimes B$ is not a skew field, hence $A \otimes B \sim C$ with some quaternion algebra C, and conversely. In Part 1 we have given a short proof of this. Hence condition (Q) is equivalent to the following condition concerning splitting fields:

> (S) *Any two non-split quaternion algebras A, B over K admit a common quadratic splitting field.*[5]

Every non-split quaternion algebra in characteristic 2 has two kinds of quadratic splitting fields: *separable* and *inseparable* ones. If the quaternion algebras A, B have a common inseparable quadratic splitting field then there is also a common separable quadratic splitting field. This seems to have first been observed by Draxl [Dra75]. A short and easy proof can be found in Lam's paper [Lam02]. Perhaps it is not without interest to note that the

[5] In today's literature, two quaternion algebras are called "linked" if they admit a common quadratic splitting field. If any two quaternion algebras over K are linked then the field K is called "linked". In this terminology, the condition (S) can be expressed by saying that K is linked.

formulas for quaternion algebras which have been used in [Lam02] are special cases for $p = 2$ of formulas which have been stated 1936 by Teichmüller for p-algebras in characteristic p [Tei36].

By the way, Arf too had all the ingredients for the proof of Draxl's theorem in his paper of 1941. For, Arf computed (on page 165 of his paper) all elements w of a quaternion algebra which generate a separable quadratic splitting field, i.e., for which $\wp(w) \in K$, but $w \notin K$.[6] This leads easily to an algorithmic construction of a common separable quadratic splitting field of two algebras if it is known that they admit a common inseparable quadratic splitting field. (See our Lemma 3 in section 5.)

But now comes the surprise: the converse does not hold. If A, B have a common separable quadratic splitting field then they do not necessarily have a common inseparable quadratic splitting field. An example can be found in Lam's paper [Lam02]. Actually, in our Part 1 too examples can be found (although at that time we were not aware of it). In fact, the quadratic forms over a rational function field $K_0(X,Y)$ which we had constructed as "counterexamples" in Part 1, have Clifford algebras with a common separable quadratic splitting field, but they do not admit a common inseparable quadratic splitting field. (This follows from our Lemma 2 in section 5.)

In view of this the following condition appears stronger than (S):

> ($\mathrm{S}_{\mathrm{ins}}$) *Any two non-split quaternion algebras A, B over K admit a common inseparable quadratic splitting field.*[7]

Clearly, (E) implies ($\mathrm{S}_{\mathrm{ins}}$) which in turn implies (S):

$$(\mathrm{E}) \Longrightarrow (\mathrm{S}_{\mathrm{ins}}) \Longrightarrow (\mathrm{S}) \Longleftrightarrow (\mathrm{Q})$$

R. Aravire and B. Jacob [AJ95] have shown that the iterated power series field $K = \mathbb{F}_2((X))((Y))$ satisfies condition (S). But it does not satisfy ($\mathrm{S}_{\mathrm{ins}}$) since there exist nonsingular anisotropic quadratic forms of dimension > 4, as we have observed already in Part 1. We see that condition ($\mathrm{S}_{\mathrm{ins}}$) is properly stronger than (S).

Condition (E) is properly stronger than ($\mathrm{S}_{\mathrm{ins}}$). For, the rational function field $K = k(X,Y)$ in two variables over an algebraically closed field k of

[6] As usual we put $\wp(X) = X^2 - X$.

[7] Perhaps K should be called "inseparably linked" if this condition is satisfied.

characteristic 2 is a C_2-field, hence every 5-dimensional quadratic form over K is isotropic. This implies that K satisfies condition ($\mathrm{S_{ins}}$); see our Lemma 2 in section 5. But its degree of imperfectness is 2 and so it does not satisfy (E).

4 The final version of Arf's first theorem

In the introduction I mentioned some "new information" which we recently received. The essential part of this news is the fact that condition ($\mathrm{S_{ins}}$) is properly stronger than (S). Once one has realized it, the question arises whether ($\mathrm{S_{ins}}$) is perhaps the adequate condition which guarantees the conclusion of Arf's Theorem 1. In fact, this is the case.

Final correction of Arf's first theorem (Baeza). *If K satisfies condition* ($\mathrm{S_{ins}}$) *then every nonsingular quadratic form over K of dimension > 4 is isotropic. Conversely, if every nonsingular quadratic form over K of dimension > 4 is isotropic then* ($\mathrm{S_{ins}}$) *holds.*
In other words: ($\mathrm{S_{ins}}$) *is necessary and sufficient for $u(K) \leq 4$.*

This theorem was proved in 1982 by Baeza [Bae82].

We recall that Arf had been looking for a condition in terms of quaternion algebras which guarantees that $u(K) \leq 4$, from which he deduced characterization of quadratic forms by their invariants. Arf's original condition (Q) turned out to be too weak and his attempt to prove his first theorem failed. Our condition (E) was sufficient but now it turns out that it is unnecessarily strong. We see that condition ($\mathrm{S_{ins}}$) is adequate because it is not only sufficient but also necessary for $u(K) \leq 4$.

Thus condition ($\mathrm{S_{ins}}$) seems to be precisely what Arf was looking for, and Baeza's Theorem above is to be regarded as the final correction and completion of Arf's theorem. In the next section we shall give a proof of Baeza's theorem.[8]

Our main aim is to put into evidence that all the ingredients of the proof are easy, and are contained in Arf's paper.

Arf could have proved Baeza's theorem if he would have realized the difference in behavior of quaternion algebras with respect to their separable and inseparable splitting fields.

[8]Another proof is contained in the thesis of Faivre in the context of a systematic theory of linkage of Pfister forms in characteristic 2: [Fai06], Prop.3.3.5.

In view of all this we may perhaps assess Arf's situation in 1940/41 as follows:

In his letter to Hasse of March 29, 1940 which we have cited already, Arf had explained that algebraic function fields of one variable over a perfect field of constants satisfy condition (E), and he showed that (E) implies (Q). In fact, what he showed in this letter was that (E) implies (S_{ins}). It is not improbable that he had a proof that (S_{ins}) implies every nonsingular quadratic form of dimension > 4 to be isotropic. In fact, as said above already all the ingredients of our proof were known to Arf.

But it appears that Arf did not realize the difference between condition (S_{ins}) and (S). Hence he started with condition (S), and since he could not prove that under this condition any two quaternion algebras have a common inseparable quadratic splitting field, he tried to use their common separable splitting field. And so, since he was convinced of his theorem due to his experience with common inseparable quadratic splitting fields, he stumbled into his error.

There are many examples in the history of mathematics which show that if people, even respected and competent mathematicians, are convinced of the validity of a theorem then they are apt to accept any decent looking proof, even at the cost of overlooking some little detail which then may necessitate a correction of the theorem.[9]

5 Proofs

As above, K denotes a field of characteristic 2. We use the following notation:

Let V be a nonsingular quadratic space of dimension 2 over K. There is a K-basis u, v of V with

$$q(u) = a\,, \quad q(v) = b\,, \quad \beta(u,v) = 1 \qquad (a, b \in K)\,. \tag{1}$$

Here, $q : V \to K$ denotes the quadratic form of V and β is the corresponding bilinear form. Let $A = C(V)$ be the Clifford algebra of V. This is a quaternion algebra over K with basis $1, u, v, w$ and the relations

$$u^2 = a\,, \quad v^2 = b\,, \quad uv + vu = 1\,, \quad w = uv\,. \tag{2}$$

[9]One of those examples is Grunwald's theorem in class field theory (1933), which was accepted by Artin, R. Brauer, Hasse and Albert (among others) until Wang presented a counterexample (1948). See, e.g., section 5 of [Roq05].

We identify the quadratic space V with the subspace of A generated by u and v, and then $q(x) = x^2$ for all $x \in V$. If $x^2 \notin K^2$ then $K(x)$ is an inseparable quadratic subfield of A. If $x^2 \in K^2$ then A splits.

We have $\wp(w) = ab \in K$. If $ab \notin \wp(K)$ then $K(w)$ is a separable quadratic subfield of A. If $ab \in \wp(K)$ then V is isotropic and A splits. Recall that $ab \bmod \wp(K)$ is the Arf invariant of V.

In this situation we show:

Lemma 1. *Let $y \in A$.*

(i) *We have $y^2 \in K$ if and only if $y \in V + K$.*

(ii) *We have $\wp(y) \in K$ if and only if $y \in V + K + w$.*

If A does not split then this lemma gives a complete description of the quadratic subfelds $L = K(y) \subset A$: If $L|K$ is inseparable then $y = x + c$ with suitable $x \in V$, $c \in K$ and then $L = K(x)$. If $L|K$ is separable then $y = x + c + w$ with $x \in V$, $c \in K$ and then $L = K(x + w)$.

Both statements **(i)** and **(ii)** can be found in Arf's paper [Arf41] on pages 161 and 165 respectively. Arf does not formulate these statements in the form of lemmas, he just performs the computations which we give in the proof below[10] and uses them in his text.

Proof of Lemma 1:

(i) We represent $y \in A$ in the form

$$y = x + z \qquad \text{with} \quad x \in V, \quad z = c_0 + c_1 w \in K(w), \quad c_0, c_1 \in K \tag{3}$$

and compute

$$\begin{aligned} y^2 &= x^2 + z^2 + xz + zx \\ &= x^2 + z^2 + c_1(xw + wx) \\ &= x^2 + z^2 + c_1 x \end{aligned} \tag{4}$$

where we have used that $xw + wx = x$ for $x \in V$, which is a consequence of the relations (2). Now, $x^2 = q(x) \in K$, $z^2 \in K(w)$ and $c_1 x \in V$. Since $V \cap K(w) = 0$ we conclude that $y^2 \in K$ if and only if $z^2 \in K$ and hence $c_1 = 0$ (since $K(w)|K$ is separable) and so $y = x + c_0 \in V + K$.

[10]Our notation is different from Arf's notation.

(ii) From (4) we obtain

$$\wp(y) = y^2 - y = x^2 - x + z^2 - z + c_1 x$$
$$= x^2 + (c_1 - 1)x + \wp(z)\,. \tag{5}$$

This is contained in K if and only if $c_1 = 1$ and so $y = x + c_0 + w \in K + V + w$. □

For later use let us make a note of the result of the above computation: *If $y = x + c + w$ with $x \in V$, $c \in K$ then*

$$\wp(y) = x^2 + \wp(c) + \wp(w) \equiv x^2 + \wp(w) \mod \wp(K)\,. \tag{6}$$

The following lemma is the key to our proof of Baeza's theorem. Let us first explain the notation. We regard K as a 1-dimensional quadratic space with the quadratic form $q(x) = x^2$ for $x \in K$. The sum $V + K \subset A$ is direct and can be regarded as the orthogonal sum $V \perp K$ of quadratic spaces.

We can also regard $K(w)$ as a quadratic space with the norm function $N : K(w) \to K$ as the corresponding quadratic form. Explicitly we have

$$N(c_0 + c_1 w) = c_0^2 + c_0 c_1 + c_1^2 \wp(w)\,. \tag{7}$$

In part (ii) of the next lemma we consider $K(w+w') \subset A \otimes A'$ as a quadratic space in the above manner, i.e., with respect to its norm function $N : K(w+w') \to K$. The sum $V + V' + K(w+w') \subset A \otimes A'$ is direct and can be regarded as the orthogonal sum $V \perp V' \perp K(w+w')$ of quadratic spaces.

Lemma 2. *Let V, V' be nonsingular quadratic spaces of dimension 2 and let A, A' be their Clifford algebras. Assume that A and A' do not split.*

(i) *A, A' have a common inseparable quadratic splitting field if and only if $V \perp V' \perp K$ is isotropic.*

(ii) *A, A' have a common separable quadratic splitting field if and only if $V \perp V' \perp K(w+w')$ is isotropic.*

Observe that this contains Draxl's theorem that the existence of an inseparable quadratic splitting field implies the existence of a separable one. For, K as a quadratic space is contained in $K(w+w')$, hence if $V \perp V' \perp K$ is isotropic then $V \perp V' \perp K(w+w')$ is isotropic too. But in the course of proof of Lemma 2 we shall have to use Draxl's theorem, so that the latter cannot be regarded as a consequence but rather as one of the sources of Lemma 2.

For the convenience of the reader we shall present a short proof of Draxl's theorem, again with the aim of putting into evidence that the ingredients of the proof can be found in Arf's paper. See Lemma 3 below.

Proof of Lemma 2:

(i) Let L be a common inseparable quadratic splitting field of A and A'. There is an embedding $L \to A$ and so we may regard L as a subfield of A. Lemma 1 shows that there exists $x \in V$ such that $L = K(x)$.

Similarly there is an embedding $L \to A'$, sending x to an element $y' \in A'$ with $y'^2 = x^2 = q(x) \in K$. By Lemma 1 again $y' = x' + c$ with $x' \in V'$ and $c \in K$. So we have

$$q(x) = q(x' + c)\,,$$

$$q(x) + q(x') + c^2 = 0\,. \tag{8}$$

Since $q(x) = x^2 \neq 0$ this shows that $V \perp V' \perp K$ is isotropic.

Conversely, assume $V \perp V' \perp K$ is isotropic. There exists a nontrivial relation $q(x+x'+c) = 0$ with $x \in V$, $x' \in V'$, $c \in K$. Regarding $V+V'+K \subset A \otimes A'$ as a subspace of $A \otimes A'$ we have

$$x^2 + x'^2 + c^2 = (x + x' + c)^2 = q(x + x' + c) = 0\,. \tag{9}$$

We may assume $x \neq 0$. Since A does not split we have $x^2 \notin K^2$, hence $K(x)$ is a quadratic inseparable subfield of A. From (9) we conclude that $K(x'+c) \subset A'$ is isomorphic to $K(x)$. Since $c \in K$ we have $K(x'+c) = K(x')$. We see that $K(x') \subset A'$ is isomorphic to $K(x) \subset A$.

(ii) Now let L be a common *separable* quadratic splitting field of A and A'. Again, there is an embedding $L \subset A$ and Lemma 1 shows that there is $x \in V$ such that $L = K(x+w)$. Similarly there is an embedding of L into A', let L' be its image. Then, again by Lemma 1 we have $L' = K(x' + w')$ with some $x' \in V'$. Since L and L' are isomorphic we have by Artin-Schreier theory:

$$\wp(x + w) \equiv \wp(x' + w') \mod \wp(K)\,, \tag{10}$$

hence using (6):

$$x^2 + \wp(w) = x'^2 + \wp(w') + \wp(c) \quad \text{with} \quad c \in K\,,$$

$$x^2 + x'^2 + \wp(c + w + w') = 0\,. \tag{11}$$

Here, $\wp(c+w+w') = N(c+(w+w'))$. (See (7) with w replaced by $w+w'$.) It follows that the quadratic space $V \perp V' \perp K(w + w')$ is isotropic.

Conversely, assume that $V \perp V' \perp K(w+w')$ is isotropic. We have to show that there exists a common separable quadratic splitting field of A and A'. There exists a nontrivial relation of the form

$$x^2 + x'^2 + N(z) = 0 \qquad \text{with} \quad x \in V,\, x' \in V',\, z \in K(w+w')\,.$$

We write $z = c_0 + c_1(w+w')$ and use formula (7) for the norm (with w replaced by $w+w'$). Thus

$$x^2 + x'^2 + c_0^2 + c_0c_1 + c_1^2\wp(w+w') = 0 \tag{12}$$

If $\wp(w) \equiv \wp(w') \bmod \wp(K)$ then the Artin-Schreier fields $K(w) \subset A$ and $K(w') \subset A'$ are isomorphic. So we may assume that $\wp(w+w') \not\equiv 0 \bmod \wp(K)$. Then $K(w+w')$ is a separable quadratic field and the norm $N : K(w+w') \in K$ is isotropic.

Suppose first that $c_1 \neq 0$. After dividing by c_1^2 on both sides in (12) and changing notation we may assume $c_1 = 1$. Hence

$$x^2 + x'^2 + \wp(c_0 + w + w') = 0$$

which gives

$$x^2 + \wp(w) = x'^2 + \wp(w') + \wp(c_0)$$

and using (6):

$$\wp(x+w) = \wp(x'+w') + \wp(c_0) \equiv \wp(x'+w') \mod \wp(K)\,.$$

It follows that the quadratic Artin-Schreier extensions $K(x+w) \subset A$ and $K(x'+w') \subset A'$ are isomorphic.

If $c_1 = 0$ then (12) shows that $V \perp V' \perp K$ is isotropic. From Part(i) of the Lemma we infer that A and A' have a common *inseparable* splitting field. We have already mentioned Draxl's theorem which says that then there is also a common *separable* splitting field. For its proof see Lemma 3 below.

□

Lemma 3 (Draxl)**.** *Let A and A' be two nonsplitting quaternion algebras over K. If A, A' have a common inseparable quadratic splitting field then they also have a common separable quadratic splitting field.*

Proof:

The statement of the lemma does not refer to particular quadratic spaces of which A and A' are Clifford algebras. So we can choose those spaces according to the situation at hand. Let $L \subset A$ and $L' \subset A'$ be isomorphic inseparable quadratic fields. We can choose the quadratic space $V = \langle u, v \rangle$ such that the first basis element u generates L, so that $L = K(u)$ with $u^2 = a \in K$. Then we choose $V' = \langle u', v' \rangle$ such that u' is the image of u under the isomorphism $L \approx L'$, and we have $u'^2 = a$. As in (2) write $v^2 = b$, $w = uv$ and similar for A'.

Now we put

$$y = cu + v \qquad \text{and} \qquad y' = cu'$$

where the coefficient $c \in K$ will be determined below. We compute, using (2)

$$y^2 = c^2 a + b + c\,, \qquad y'^2 = c^2 a\,.$$

If the coefficient c is chosen such that $c = \wp(w') - \wp(w) - b$ then we obtain in view of (6):

$$\begin{aligned} \wp(y + w) &= y^2 + \wp(w) \\ &= y'^2 + \wp(w') \\ &= \wp(y' + w') \end{aligned}$$

We see that the separable quadratic Artin-Schreier fields $K(y + w) \subset A$ and $K(y' + w')$ are isomorphic. □

Proof of Baeza's theorem (see section 4):

(i) Suppose K satisfies condition ($\mathrm{S}_{\mathrm{ins}}$). Consider a nonsingular quadratic space of dimension > 4 and write it in the form

$$V \perp V' \perp W$$

where V and V' are of dimension 2 and W of dimension > 0. We have to show that $V \perp V' \perp W$ is isotropic.

Let $0 \neq c \in K$ and consider the scaled spaces $V^{(c)}$, $V'^{(c)}$, $W^{(c)}$ with the scaling factor c. Isotropy of $V \perp V' \perp W$ is equivalent to isotropy of the scaled space $V^{(c)} \perp V'^{(c)} \perp W^{(c)}$. We take $c = q(y)^{-1}$ where $y \in W$ is chosen such that $q(y) \neq 0$. We have $c\,q(y) = 1$. Thus Ky (as a 1-dimensional quadratic space in the scaled space $W^{(c)}$) is isomorphic to K. Lemma 2(i) now shows that $V^{(c)} \perp V'^{(c)} \perp Ky$ is isotropic. Hence $V^{(c)} \perp V'^{(c)} \perp W^{(c)}$ is isotropic too, and so is $V \perp V \perp W$.

(ii) Conversely we assume that every nonsingular space of dimension > 4 is isotropic. Let A, A' be two non-split quaternion algebras over K. We have to show that A, A' have a common inseparable quadratic splitting field.

We write $A = C(V)$ as the Clifford algebra of a 2-dimensional nonsingular quadratic space V as in (1), (2), and similarly $A' = C(V')$. As above we consider the separable quadratic extension $K(w + w') \subset A \otimes A'$ as a quadratic space with respect to the norm. The 6-dimensional space $V \perp V' \perp K(w + w')$ is isotropic, and hence Lemma 2(ii) shows that A and A' have a common separable quadratic splitting field. But we are looking for a common *inseparable* quadratic splitting field; this will be established as follows.

First, the common separable quadratic splitting field can be embedded into A and into A'; this yields isomorpic separable quadratic subfields in A and in A'. After changing notation we now assume that the fields $K(w) \subset A$ and $K(w') \subset A'$ are isomorphic. We identify $K(w) = K(w')$ so that A, A' appear as crossed products of the same separable quadratic field $K(w)$, i.e., $A = (a, K(w)\,]$ and $A' = (a', K(w)\,]$, in the notation introduced by Teichmüller [Tei36].

Second, we represent the quadratic spaces V, V' as norm spaces in the following manner: We have already said above that $K(w)$ can be regarded as a quadratic space with respect to the norm function $N : K(w) \to K$. We scale the norm by the factor a and thus consider the scaled space $K(w)^{(a)}$ with the quadratic form $aN : K(w) \to K$. It is well known that there is an isomorphism as quadratic spaces:

$$V \xrightarrow[\approx]{\varphi} K(w)^{(a)} \,. \tag{13}$$

For, if we write $x \in V$ in the form $x = x_1 u + x_2 v$ with $x_1, x_2 \in K$ then we compute

$$\begin{aligned} q(x) = x^2 &= a x_1^2 + x_1 x_2 + b x_2^2 \\ &= a(x_1^2 + x_1 \overline{x}_2 + ab\overline{x}_2^2) \qquad \text{with} \qquad \overline{x}_2 = a^{-1} x_2 \\ &= a(x_1^2 + x_1 \overline{x}_2 + \wp(w)\overline{x}_2^2) \\ &= aN(x_1 + \overline{x}_2 w) \\ &= aN\varphi(x) \end{aligned} \tag{14}$$

if we put

$$\varphi(x) = x_1 + \overline{x}_2 w \,.$$

The formula (14) which expresses the isomorphism (13) can be found in Arf's paper on page 153 where he uses it for the proof of his *Satz 3*.

Thus we may replace V by the quadratic space $K(w)^{(a)}$ and similarly V' by $K(w)^{(a')}$. By hypothesis, the 6-dimensional space

$$K(w)^{(a)} \perp K(w)^{(a')} \perp K(w)$$

is isotropic. Thus there is a nontrivial relation of the form

$$aN(x) + a'N(x') + N(y) = 0 \qquad \text{with} \qquad x,\, x',\, y \in K(w)\,. \tag{15}$$

If $y \neq 0$ then after dividing by $N(y)$ and changing notation we obtain

$$aN(x) + a'N(x') + 1 = 0 \qquad \text{with} \qquad x, x' \in K(w)\,.$$

This shows that $K(w)^{(a)} \perp K(w)^{(a')} \perp K$ is isotropic, i.e., $V \perp V' \perp K$ is isotropic. Now Lemma 2(i) shows that there exists a common inseparable quadratic splitting field of A and A'.

If $y = 0$ then from (15) we infer that $V \perp V'$ is isotropic, hence $V \perp V' \perp K$ is isotropic too and again Lemma 2(i) applies. □

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