# 5

# The Time Arrow of Spacetime Geometry

In the framework of general relativity, gravity is a consequence of spacetime curvature. Its dynamical laws (Einstein's field equations) are again symmetric under time reversal. However, if their *actual* global solution, that is, the observed spacetime, is asymmetric (such as a forever expanding universe), this must affect the dynamics of all matter. While this was well known, it came as a surprise during the early 1970s that strongly gravitating systems possess thermodynamical properties, thus indicating an intimate connection between two seemingly very different fields of physics.

Gravitating systems are already thermodynamically peculiar in Newton's theory, since they possess negative heat capacity, resulting from the universal attractivity of this force. In particular, attractive forces which depend homogeneously on the minus second power of distance, such as gravity and Coulomb forces, lead according to the virial theorem to the relation

$$\overline{E_{\mathrm{kin}}} = -\frac{1}{2}\overline{E_{\mathrm{pot}}} = -E \ , \tag{5.1}$$

between the mean values of kinetic and potential energies, and therefore between them and the total energy. This virial theorem is valid for mean values over a (quasi-)period of the motion, or approximately (in the case of semistable states) for mean values defined over sufficiently large intervals of time. In quantum theory, mean values have to be replaced by expectation values on proper (normalizable) energy eigenstates. The theorem can then be conveniently proved using Fock's *ansatz* $\psi(\lambda\boldsymbol{r}_1, \ldots, \lambda\boldsymbol{r}_N)$ and the homogeneity of $T$ and $V$ in a variational procedure, $\delta\big(\langle\psi|T+V|\psi\rangle/\langle\psi|\psi\rangle\big) = 0$, with respect to $\lambda$. So it must also hold for expectation values on density matrices whose non-diagonal elements can be neglected in the energy basis. (For relativistic generalizations of the virial theorem see Gourgoulkon and Bonazzola 1994.)

The anti-intuitive negative sign relating kinetic and total energy in (5.1) means, for example, that satellites are *accelerated* by friction when they enter the earth's atmosphere, and that stars *heat up* by radiating energy away. This second example is valid only as far as the quantum mechanical zero-point energy does not dominate $\overline{E_{\mathrm{kin}}} = \mathrm{Trace}\{\rho T\}$ – as it would in white dwarf stars or solid bodies. Early astrophysicists believed instead that stars always cool down in the course of time. The virial theorem also means that the heat flow from hot to cold objects which are governed by gravity causes a thermal inhomogeneity to *grow*.

To construct an example, first consider a monatomic ideal gas in two vessels under different conditions, but under exchange of energy (heat), $\delta U_1 = -\delta U_2$, and particles, $\delta N_1 = -\delta N_2$. Their partial entropies according to (3.14) are given by

$$S_i = kN_i \left( \frac{3}{2} \ln T_i - \ln \rho_i + C \right) , \tag{5.2}$$

with $i = 1, 2$ distinguishing the two vessels. Since the internal energy, $U = \overline{E_{\mathrm{kin}}}$, is here $U = (3/2)NkT$, the total change of entropy becomes for fixed volumes $V_i$, or for fixed densities $\rho_i = N_i/V_i$,

$$\delta S_{\mathrm{total}} = \delta S_1 + \delta S_2 = \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \delta U_1 + k \left( \frac{3}{2} \ln \frac{T_1}{T_2} - \ln \frac{\rho_1}{\rho_2} \right) \delta N_1 \; . \tag{5.3}$$

This expression describes entropy changes $\delta S_1$ and $\delta S_2$ with opposite signs, which cancel only in thermodynamical equilibrium ($T_1 = T_2$ and $\rho_1 = \rho_2$). In this situation without gravity, an entropy increase in accordance with the Second Law requires a *reduction* of thermal and density inhomogeneities (except for the transient *thermo-mechanical effect*, that is, a thermally induced pressure difference that is caused by the temperature dependence of the second term).

However, the density of a gravitating star is not a free variable that can be kept fixed (as in the laboratory). A typical star, assumed for simplicity to be in thermal equilibrium, may to a very good approximation also be described as an ideal gas. Its temperature and volume are then related by means of the virial theorem according to

$$NT \propto U = \overline{E_{\mathrm{kin}}} \propto -\overline{E_{\mathrm{pot}}} \propto \frac{N^2}{R} \propto \frac{N^2}{V^{1/3}} \; , \tag{5.4}$$

that is, $V \propto N^3/T^3$. The entropy (5.2) of a star is therefore

$$\begin{aligned} S_{\mathrm{star}} &= kN \left( \frac{3}{2} \ln T - \ln N + \ln V + C \right) \\ &= kN \left( -\frac{3}{2} \ln T + 2 \ln N + C' \right) \; . \end{aligned} \tag{5.5}$$

In the second line, the signs of $\ln T$ and $\ln N$ are reversed. The total entropy change of a star embedded in an interstellar gas, $\delta S_{\mathrm{star}} + \delta S_{\mathrm{gas}}$, becomes after again using the virial theorem in the form $E_{\mathrm{star}} = -U_{\mathrm{star}}$ ,

$$\delta S_{\rm total} = \left(\frac{1}{T_{\rm star}} - \frac{1}{T_{\rm gas}}\right)\delta E_{\rm star} + k\ln\left[\frac{C'' N_{\rm star}^2 \rho_{\rm gas}}{(T_{\rm star}T_{\rm gas})^{3/2}}\right]\delta N_{\rm star} \; . \tag{5.6}$$

While heat must still flow from the hot star into cold interstellar space in order to comply with the Second Law, this leads now to a further increase of the star's temperature, and the accretion of matter – provided the 'star' is already sufficiently massive. Thermal and density inhomogeneities thus *grow* in the generic astrophysical situation, although there are also 'pathological' objects with non-periodic motion, such as gravitationally collapsing spherical matter shells or pressure-free dust spheres, for which the virial theorem does not hold.

These arguments show that the evolution of normal stars is dynamically controlled by thermodynamics rather than by gravity itself. If the thermodynamical arrow of time did change direction in a recontracting universe (as suggested by Gold 1962 – see Sect. 5.3), stars and other gravitating objects would have to re-expand by means of advanced incoming radiation in spite of their attractive forces.

A homogeneous universe must therefore describe an unstable state of very low entropy (though a 'simple' state in the sense of Sect. 3.5). So one may ask whether the evolution of matter into inhomogeneous clumps under gravitational forces represents an entropy capacity that is sufficient to explain the observed global thermodynamical arrow of time. The apparently required *Kaltgeburt* of the Universe might then be replaced by a *homogeneous birth*, since inhomogeneous local contraction leads to the formation of strong temperature and density gradients.

In order to estimate the improbability (negentropy) of a homogeneous universe, one has to know the maximum entropy that can be gained by gravitational contraction. Conceivable limits of contraction are:

- *Quantum degeneracy* (primarily of electrons) is essential for the stability of solid gravitating bodies and white dwarf stars. By emitting heat, these objects cool down rather than further heating up.
- *Repulsive short range forces* are important in neutron stars, for example.
- *Gravitation* itself may lead to black holes even in Newton's theory. Any radiation with bounded velocity cannot escape from the surface of a sufficiently dense and massive object. If this velocity bound is as universal as gravity (as in the theory of relativity), the further fate of matter inside this critical surface remains completely *irrelevant* to an external observer. This surface defines an *event horizon* for him. Matter disappearing behind the horizon is irreversibly lost except for its long range forces, such as gravity itself. In particular, it can no longer participate in the thermodynamics of the Universe.

Such *non-relativistic black holes* were discussed by Laplace as early as 1795, and before him by J. Mitchel at Cambridge. In general relativity, black holes are described by specific spacetime structures. This leads to the further consequence that neither of the first two mentioned limits to gravitational contraction may prevent an object of sufficiently large mass (that could always be reached by further accretion of matter) from collapsing into a black hole. Repulsive forces would give rise to a positive potential energy, that must eventually dominate as a source of gravity, while the increasing zero point pressure of a degenerate Fermi gas would force the fermions into effective bosons that may form a further contracting condensate.

Therefore, only black holes define a realistic upper limit for entropy production by gravitational contraction of matter from the point of view of an external observer. But what is the value of the entropy of a black hole? This question cannot be answered by investigating relativistic stars, that is, equilibrium systems, since the essential stages of the collapse proceed irreversibly. However, a unique and finite answer is obtained from a quantum aspect of black holes, viz., their Hawking radiation (Sect. 5.1).

Since in general relativity the spatial curvature represents a dynamical state (see Sect. 5.4), it may itself carry entropy. Its dynamics is described by Einstein's field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{5.7}$$

in units with $G = c = 1$, where $T_{\mu\nu}$ is the energy–momentum tensor of matter. They define an initial (or final) value problem, since they are essentially of hyperbolic type (see Sect. 2.1). The Einstein tensor $G_{\mu\nu}$ is a linear combination of the components of the Ricci tensor $R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$, that is, the trace of the Riemann curvature tensor. Forming this trace is analogous to forming the d'Alembertian in the wave equation (2.1) for the electromagnetic potential from its matrix of second derivatives $\partial_\nu \partial_\lambda A^\mu$. Aside from nonlinearities (that are responsible for the self-interaction of gravity), the Riemann curvature tensor is similarly defined by the second derivatives of the metric $g_{\mu\nu}$, which thus assumes the role of the gravitational potential (analogous to $A^\mu$ in electrodynamics). In both cases, the trace of the tensor of derivatives is determined locally by the sources, while its trace-free parts represent the degrees of freedom of the vector or tensor field, respectively, which can therefore be freely *chosen* initially (as an incoming field).

Penrose (1969, 1981) used this freedom to conjecture that the trace-free part of the curvature tensor (the *Weyl tensor*) vanished when the Universe began. This situation describes a 'vacuum state of gravity', that is, a state of minimum gravitational entropy, and a space as flat as is compatible with the sources. It is analogous to the cosmic initial condition $A^\mu_{\text{in}} = 0$ for the electromagnetic field discussed in Sect. 2.2 (with Gauss's law as a similar constraint). Gravity would then represent a retarded field, requiring 'causes' in the form of advanced sources. Since Penrose intends to explain the thermodynamical arrow, too, from this initial condition (see Sect. 5.3), his conjecture revives Ritz's position in his controversy with Einstein (see Chap. 2) by applying it to gravity rather than to electrodynamics.

In the big bang scenario, the beginning of the Universe is characterized by a past time-like curvature singularity (where time itself began). Penrose used this fact to postulate his Weyl tensor hypothesis on all past singularities, since this would allow only *one* of them: a uniform big bang. In the absence of an *absolute* direction of time, the past would then be distinguished from the future precisely and solely by this asymmetric boundary condition and its consequences (again introducing a 'double standard'). If the Weyl tensor condition could be derived from some other assumptions that did *not* arbitrarily select a time direction, it would have to exclude inhomogeneous future singularities as well. This may again lead to dynamical consistency problems, but it would not rule out collapsing objects to *appear* as black holes to external observers (see Sects. 5.1 and 6.2.3).

## 5.1 Thermodynamics of Black Holes

In order to discuss the spacetime geometry of black holes, it is convenient to consider the static and spherically symmetric vacuum solution, discovered by Schwarzschild and originally expected to represent a point mass. In terms of spherical spatial coordinates, this solution is described by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ . \tag{5.8}$$

Here, $r$ measures the size of a two-dimensional sphere − though *not* the distance from $r = 0$. This metric form is singular at $r = 0$ and $r = 2M$, but the second singularity, at the *Schwarzschild radius* $r = 2M$, is merely the result of an inappropriate choice of these coordinates. The condition $r = 2M$ describes a surface of fixed area $A = 4\pi(2M)^2$ (using Planck units $G = c = \hbar = k_{\mathrm{B}} = 1$) in spite of moving outwards at speed of light. In its interior (that is, for $r < 2M$) one has $g_{tt} = 2M/r - 1 > 0$ and $g_{rr} = (1 - 2M/r)^{-1} < 0$. Therefore, $r$ and $t$ interchange their physical meaning as spatial and temporal coordinates. This internal solution is *not* static, while the genuine singularity at $r = 0$ represents a time-like singular boundary rather than the space point expected by Schwarzschild.

Physical (time-like or light-like) world lines, that is, curves with $\mathrm{d}s^2 \leq 0$, hence with $(\mathrm{d}r/\mathrm{d}t)^2 \leq (1 - 2M/r)^2 \to 0$ for $r \to 2M$, can only approach the Schwarzschild radius parallel to the $t$-axis (see Fig. 5.1). Therefore, the interior region $r < 2M$ is physically accessible only via $t \to +\infty$ or $t \to -\infty$, albeit within finite proper time. These world lines can be extended regularly into the interior when $t$ goes beyond $\pm\infty$. Their proper times continue into the physically finite future (for $t > +\infty$) or past (for $t < -\infty$) with the new time coordinate $r < 2M$. There are therefore *two* internal regions (II and IV in the figure), with their own singularities at $r = 0$ (at a finite distance in proper times). These internal regions must in turn each have access to a new *external*