

where  $\{P_n\}$  (no caret!), with  $P_m P_n = P_m \delta_{mn}$ , is a complete set of projection operators on mutually orthogonal subspaces of the Hilbert space of quantum states. In quantum field theory, projections on ‘unitarily inequivalent’ separable subspaces of Hilbert space, sometimes even regarded as ‘distinct Hilbert spaces’, are often chosen for this purpose. However, these decompositions of non-separable Hilbert spaces are no less arbitrary than any other  $\hat{P}_{\text{semidiag}}$  (though often *useful* in the case of large numbers of effective degrees of freedom). If imposed axiomatically, the relevance concept (4.11) may represent a *superselection rule* (Wick, Wightman and Wigner 1952, Jauch 1968, Hepp 1972). This observation suggests that proposed superselection rules are similarly based on some *dynamical* robustness like the ‘thermodynamically macroscopic’ variables of Chap. 3 that are usually assumed as ‘given’ – a possibility that will be further investigated and confirmed in Sect. 4.3.

#### 4.1.2 Master Equations and Quantum Indeterminism

The Hamiltonian of a quantum mechanical system is often written in the form  $H = H_0 + H_1$  in order to derive a master equation in terms of a perturbation expansion with respect to  $H_1$ . However, the main purpose of this split Hamiltonian is to define a relevance concept of type (4.9) or (4.11) by means of the eigenbasis of  $H_0$ . It may then (but need not) be *further* used for a time-dependent perturbation expansion with respect to the off-diagonal elements of  $H$  in this representation.

The dynamics of the ‘relevant’ part  $\hat{P}_{\text{diag}}\rho$  is the dynamics of the diagonal elements of  $\rho$ . According to (4.1) one has in any representation (now writing  $\hat{P}_{\text{diag}} = \hat{P}$  for short)

$$\begin{aligned} i \frac{d\rho_{mm}}{dt} &= \sum_n (H_{mn}\rho_{nm} - \rho_{mn}H_{nm}) \\ &\equiv \sum_{n(\neq m)} (H_{mn}\rho_{nm} - \rho_{mn}H_{nm}) \hat{=} \hat{P}\hat{L}(1 - \hat{P})\rho. \end{aligned} \quad (4.12)$$

Since the diagonal matrix elements of  $\rho$  do not contribute to the RHS, the first term of Zwanzig’s pre-master equation (3.44), representing  $\hat{P}\hat{L}\hat{P}$ , vanishes for this relevance concept. The terms remaining in (4.12) describe the coupling to the ‘irrelevant’ off-diagonal elements, and demonstrate that the diagonal elements are dynamically autonomous only in the trivial case (see footnote 6 of Chap. 3 regarding the quantum Zeno effect). Because of the formal analogy, the rest of Zwanzig’s method can then be applied, provided the required approximations are valid. The propagator  $\exp[-i(1 - \hat{P})\hat{L}\tau]$ , occurring in the operator  $\hat{G}_{\text{ret}}$  of the Markovian approximation (3.48), defines here a closed but highly non-trivial dynamics of the off-diagonal elements of  $\rho_{mn}$ .

Pauli’s master equation can now be obtained from (3.48) and (3.45) by using a perturbation expansion in terms of the off-diagonal elements of the

Hamiltonian for calculating  $\hat{G}_{\text{ret}} = \int_0^T \hat{G}(\tau) d\tau$ . These off-diagonal elements are thus assumed to be small, although the master equation would become trivial if they vanished exactly (that is, for  $H = H_0$ ). This last remark emphasizes the *dynamical* role of the relevance concept.

Now consider the last three factors of the RHS of the integral kernel (3.45) applied to  $\rho$ :

$$(1 - \hat{P})\hat{L}\hat{P}\rho = (1 - \hat{P})[H, \hat{P}\rho] \hat{=} H_{mn}(\rho_{nn} - \rho_{mm}) \quad \text{with } m \neq n. \quad (4.13)$$

This expression depends only on the off-diagonal elements of  $H$ . The projection  $1 - \hat{P}$  is ineffective, as  $\hat{P}\hat{L}\hat{P} = 0$ . Similarly, one has for the *first* three factors of the RHS of (3.45), when applied to any matrix  $X$ :

$$\hat{P}\hat{L}(1 - \hat{P})X \hat{=} \sum_{k(\neq m)} (H_{mk}X_{km} - X_{mk}H_{km}). \quad (4.14)$$

Hence,  $\hat{G}_{\text{ret}}$  is of second and higher orders in the off-diagonal elements of  $H$ . When neglecting higher orders according to Pauli, one has to express the remaining propagator  $\exp[-i(1 - \hat{P})\hat{L}\tau]$  in (3.45) solely in terms of diagonal elements of  $H$ ,  $H_{mm} =: E_m^{(0)}$ . This means

$$e^{-i(1 - \hat{P})\hat{L}\tau} X \hat{=} e^{-i(E_m^{(0)} - E_n^{(0)})\tau} X_{mn}, \quad (4.15)$$

and one obtains

$$\begin{aligned} \hat{P}\hat{L}(1 - \hat{P})e^{-i(1 - \hat{P})\hat{L}\tau}(1 - \hat{P})\hat{L}\hat{P}\rho \hat{=} \\ \sum_n |H_{mn}|^2 2 \cos[(E_m^{(0)} - E_n^{(0)})\tau](\rho_{mm} - \rho_{nn}). \end{aligned} \quad (4.16)$$

This result corresponds to a Born approximation in terms of the off-diagonal elements of the Hamiltonian. The time integral required to obtain  $\hat{G}_{\text{ret}}$  according to (3.49) leads to the resonance factor

$$\int_0^T \cos[(E_m^{(0)} - E_n^{(0)})\tau] d\tau = \frac{\sin[(E_m^{(0)} - E_n^{(0)})T]}{(E_m^{(0)} - E_n^{(0)})}, \quad (4.17)$$

familiar from time-dependent perturbation theory. In the limit  $T \rightarrow \infty$ , this quotient becomes a  $\delta$ -function times  $\pi$ , and (3.48) can be written (Pauli 1928)

$$\frac{d\rho_{mm}}{dt} = 2\pi \sum_n |H_{mn}|^2 \delta(E_m^{(0)} - E_n^{(0)})(\rho_{nn} - \rho_{mm}) =: \sum_n A_{mn}(\rho_{nn} - \rho_{mm}). \quad (4.18)$$

This *Pauli equation* is similar to other master equations, such as (3.51), while the coefficients  $A_{mn}$ , defined on the RHS, are transition rates in analogy to Boltzmann's  $w(\mathbf{p}_1\mathbf{p}_2, \mathbf{p}'_1\mathbf{p}'_2)$  of Sect. 3.1.1. If  $H_1$  contains only two-particle

interactions, the sum over  $n$  may indeed be written as a sum over particle pairs. According to the above definition, the coefficients  $A_{mn}$  conserve energy and satisfy the symmetry under collision inversion,  $A_{mn} = A_{nm}$  [see (3.7)]. Therefore, the Pauli equation conserves total probability,  $\sum_n d\rho_{nn}/dt = 0$ .

The explicit form of the Pauli equation (4.18) may be used to discuss its range of validity, which must be limited by the approximations used when deriving the general master equation (3.48). It depends here on the spectrum of the Hamiltonian, which is often discrete for quantum systems. Nonetheless, Poincaré recurrence times can be neglected *in practice* for macroscopic quantum systems. Their energy spectra are usually so dense that they do *not* lead to any observable differences compared to a continuous spectrum. Quantum systems may even exhibit ‘classical chaos’ (Habib, Shizume and Zurek 1998). On the other hand, even a continuous spectrum would not by itself justify an arrow of time (as is often claimed). The negligibility of recurrences for all times of interest – whether they exist in principle or not – applies in *both* directions of time. The physical importance of the difference between discrete and continuous spectra seems to be grossly overemphasized in mathematical foundations of irreversibility.

However, the energy  $\delta$ -function occurring in (4.18) is meaningful only inside an integral over energy  $E$ , or, as an approximation, under a sum over  $m$ . Therefore, Pauli combined groups of states with almost equal energies to form ‘cells’ (subspaces) representing a coarse-graining in order to apply a random phase approximation in the corresponding sums (see also van Kampen 1954). Erich Joos (1984) was able to show that the off-diagonal elements  $\rho_{mn}$  between states from such macroscopically different subspaces disappear by interaction with the environment (‘decoherence’ – see Sect. 4.3). This *dynamical* argument justifies Pauli’s conceptual cells and his random phase ‘approximation’.

When applied to a single initial state with  $\rho_{00}(0) = 1$ , Pauli’s equation (4.18) assumes the form of *Fermi’s Golden Rule* in the Born approximation. Replacing the sum over initial states  $n$  in (4.18) by an energy integral and a sum over all remaining quantum numbers  $\beta$ , that is,  $\sum_n \dots \longrightarrow \sum_\beta \int \sigma_\beta(E) \dots dE$  with a partial density of states  $\sigma_\beta(E)$ , and similarly substituting  $m \longrightarrow E', \alpha$  for the final states, one obtains for the energy-integrated diagonal elements of final states  $\alpha \neq 0$ ,  $\rho_{\alpha\alpha} := \int \rho_{E'\alpha, E'\alpha} \sigma_\alpha(E') dE'$ :

$$\frac{d\rho_{\alpha\alpha}}{dt} = 2\pi |H_{\alpha 0}(E)|^2 \sigma_\alpha(E) \quad . \quad (4.19)$$

Here,  $H_{\alpha 0}(E) := H_{\alpha E, 0E}$ , while  $\alpha$  represents a ‘decay channel’.

Although this Golden Rule (4.19) can thus be derived as an approximation from the unitary dynamics (4.12), it is mainly used to calculate *probabilities* for decay and other non-unitary ‘quantum events’ – conventionally described by a collapse of the wave function – see Sect. 4.6. (*Coherent* exponential decay according to the Schrödinger equation will be discussed in Sect. 4.5.) In contrast, Boltzmann’s probabilistic transition rates  $w(\mathbf{p}_1 \mathbf{p}_2, \mathbf{p}'_1 \mathbf{p}'_2)$  refer to *ensembles of individually deterministic* collision trajectories (distinguished by their

impact parameters). This different interpretation is facilitated by the fact that the formal concept of a density operator is already based on a probability interpretation (see Sect. 4.2). Nobody has ever been able to construct a model that would consistently explain the wave function as representing an ensemble of ‘hidden variables’. (Bohm’s theory, that *presumes* Schrödinger’s wave function, will be discussed in Sect. 4.6.) In particular, the entropy (4.4) does not contain any contribution that might represent the missing information corresponding to such an ensemble (as in Fig. 3.5 for classical measurements).

Pauli’s equation does indeed resemble Born’s original formulation of the probability interpretation (Born 1926). Born used it to describe ‘quantum jumps’ between Schrödinger’s stationary eigenstates of Hamiltonians  $H_0$  that characterize isolated microscopic systems (such as atoms).<sup>1</sup> In quantum field theory, a similar splitting of the Hamiltonian is used to define the *interaction picture*. The special role attributed to the eigenstates of  $H_0$  as representing the ‘real’ physical states, dynamically connected by discrete jumps, was historically motivated by their correspondence with Bohr’s discrete atomic electron orbits. Quantum jumps (or a ‘collapse of the wave function’) are, of course, incompatible with deterministic *trajectories in Hilbert space*, that is, with time-dependent wave functions evolving according to a Schrödinger equation. The system Hamiltonians  $H_0$  are thus assumed not to contain any interaction that would be responsible for stochastic transitions. This early attempt to objectivize the probability interpretation (or the observables used therein) by a dynamical process is therefore based on an essential approximation. (Recall the trivial result obtained for the Pauli equation in the exact energy basis!)

The general structure of the Pauli equation is preserved even when the perturbation expansion in terms of the off-diagonal elements of  $H$  (in a certain basis) is not used. This improved equation is known as Van Hove’s ‘exact’ master equation (Van Hove 1957). It represents the master equation for the Zwanzig projection (4.9) without any *further* approximation. In particular, if the chosen basis of relevance (the eigenbasis of  $H_0$ ) is the independent particle basis, the matrix elements  $H_{mn}$  appearing in the Pauli equation have to be replaced by the elements of a  $T$ -matrix, usually defined as  $T := (S - 1)/2\pi i$ , where  $S$  is the exact two-particle scattering matrix. This procedure presumes the negligibility of simultaneous many-particle collisions (just as Boltzmann’s *Stoßzahlansatz*). However, the adjective ‘exact’ for Van Hove’s equation is misleading even for a dilute gas, as it refers only to the calculation of  $\hat{G}_{\text{ret}}$ ,

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<sup>1</sup> While Born may not have been using his concepts quite consistently in these early days of quantum mechanics, in his third (here quoted) paper on the probability interpretation he discussed probabilities for jumps between stationary *wave functions* – not probabilities for the occurrence of classical properties (such as particle positions). In scattering or decay ‘events’ he referred to plane waves as stationary states, which he then *associated* with particle momenta according to de Broglie’s relation. One year before the formulation of the uncertainty relations this was not recognized as being in conflict (in principle) with the position measurement at the detector.

but not to the derivation of the master equation (3.48) in its preferred basis of relevance. Similarly to the choice of subspaces in (4.11), Born's probability interpretation, when applied to *measurements*, depends on the choice of appropriate 'observables'.

In analogy to the classical  $H$ -theorem (3.10), one may again show that the entropy corresponding to the Zwanzig projection  $\hat{P}_{\text{diag}}$  never decreases under the Pauli or Van Hove equation:

$$\frac{dS[\hat{P}_{\text{diag}}\rho]}{dt} = -k \frac{d\left(\sum \rho_{mm} \ln \rho_{mm}\right)}{dt} \geq 0. \quad (4.20)$$

Evidently, this entropy depends crucially on the chosen basis for diagonalization, that is, on the specific concept of relevance used in this master equation.

Because of the formal analogy, the classical canonical distribution,  $\rho_{\text{can}}(p, q) = Z^{-1} \exp[-H(p, q)/kT]$ , now becomes a canonical density operator,  $\rho_{\text{can}} = Z^{-1} \exp(-H/kT)$ . It can be derived precisely as in (3.19) by maximizing the entropy  $S[\rho]$  under the constraint of fixed mean energy and probability norm. The so-called 'new statistics' (Bose or Fermi statistics) in terms of apparent *particles* is obtained when evaluating this canonical density operator in terms of quantum states of free *fields* – conveniently in the *occupation number representation*. Only when expressed in terms of particle states does it appear as a new method for counting them. The success of quantum statistics is indeed one of the strongest arguments against *particles* (in their original sense of pointlike objects in space, distinguishable by their trajectories) as a fundamental kinematical concept.

This conclusion, that fields rather than particles have to be quantized even for fields that never appear classically (such as spinor fields – see Zeh 2003), is also supported by the absence of Gibbs' self-mixing entropy (see footnote 2 of Chap. 3). The empirically correct measure on phase space,  $d^{3N}p d^{3N}q/h^{3N}N!$ , may then be obtained, for example, in the partition function  $Z$  for a grand canonical ensemble,  $p_{E,N}(\mu, T) = \exp[-(E - \mu N)/kT]$ . If this expression is evaluated by means of the familiar textbook approximation in the occupation number representation  $|\{n_{\mathbf{k}}\}\rangle$  for spatial wave modes (often incorrectly regarded as 'single-particle' *wave functions*) with wave numbers  $\mathbf{k} = \mathbf{p}/\hbar$  on a large space volume  $V$ , one obtains for dilute gases – where  $N = \sum_{\mathbf{k}} n_{\mathbf{k}}$  and  $E = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}}$ , with  $\varepsilon_{\mathbf{k}} = p(\mathbf{k})^2/2m$  and  $\varepsilon_{\mathbf{k}} - \mu \gg kT$ :

$$\begin{aligned} Z(\mu, T) &= \sum_{\{n_{\mathbf{k}}\}} \exp\left[-\sum_{\mathbf{k}} \frac{(\varepsilon_{\mathbf{k}} - \mu)n_{\mathbf{k}}}{kT}\right] \\ &\approx \sum_N \frac{V^N}{h^{3N}N!} \exp\left(\frac{N\mu}{kT}\right) \int \exp\left(-\sum_{i=1}^N \frac{\mathbf{p}_i^2}{2mkT}\right) d^{3N}p \\ &\approx \sum_N \left[\frac{V}{h^3N} \exp\left(\frac{\mu}{kT}\right) \int \exp\left(-\frac{\mathbf{p}^2}{2mkT}\right) d^3p\right]^N. \end{aligned} \quad (4.21)$$

The factorials  $N! \approx N^N$  in the denominator are here *required* (as already known to Planck in 1900) in order to compensate for the sum over all permutations of the  $N$  momenta  $\mathbf{p}_i$  in this  $N$ -fold integral, since they all represent the *same* oscillator quantum states for the various wave modes. The latter are described by wave numbers  $\mathbf{k}$  which formally correspond to momenta  $\mathbf{p}$ . The density matrix, and therefore the partition function, now factorize in terms of wave modes  $\mathbf{k}$  rather than in terms of particle numbers, while the factorials do not have to be introduced ad hoc (as done by Satyendra Nath Bose in order to justify his photon concept).

**General Literature:** Jancel 1963.

## 4.2 Ensembles Versus Entanglement

Quantum wholeness is analyzable.

In the previous section, we derived the von Neumann equation from the Liouville equation by using the formal quantization rules. The dynamics of the density matrix, obtained in this way, is unitary. Therefore, it conserves  $S[\rho]$ , while the Pauli (or Van Hove) equation, albeit apparently derived from the von Neumann equation *as an approximation*, may seem to be superior, as it is able to describe quantum indeterminism and an increase in ensemble entropy, in particular in quantum measurements.

The Liouville equation itself was obtained in Sect. 3.1.2 by applying Hamilton's (that is, Newton's) equations to ensembles that represent incomplete knowledge about classical states. Since quantization of the Hamiltonian dynamics of mechanical systems leads to the Schrödinger equation, one may as well first quantize and then consider ensembles of its solutions  $\psi_\alpha(t)$  with corresponding probabilities  $p_\alpha$ , now describing incomplete knowledge *about the wave function* (see Fig. 4.1). This procedure may offer deeper insight into the meaning of the density matrix than its formal foundation of Sect. 4.1.1.

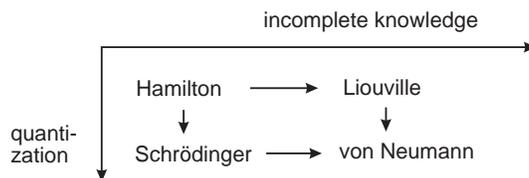
According to this ensemble interpretation, probabilities  $p_\alpha$  rather than the density matrix  $\rho(q, q')$  correspond *conceptually* to the probability distribution  $\rho_\Gamma(p, q)$ . The meaning of the density matrix can only be appreciated when considering ensemble expectation values of observables  $A$ , that is, *mean values of expectation values* with respect to different wave functions  $\psi_\alpha$ :

$$\langle A \rangle := \sum_{\alpha} p_{\alpha} \langle \psi_{\alpha} | A | \psi_{\alpha} \rangle = \text{Trace}\{A\rho\} = \sum_n a_n \langle \phi_n | \rho | \phi_n \rangle, \quad (4.22)$$

with

$$A := \sum_n |\phi_n\rangle a_n \langle \phi_n| \quad \text{and} \quad \rho := \sum_{\alpha} |\psi_{\alpha}\rangle p_{\alpha} \langle \psi_{\alpha}|.$$

The symbol  $\langle A \rangle$  denotes here a twofold mean: with respect to the ensemble of quantum states  $\psi_\alpha$  with their probabilities  $p_\alpha$ , *and* with respect to



**Fig. 4.1.** Two routes from classical mechanics to the von Neumann equation

the quantum mechanical indeterminism of measurement results  $a_n$  with their probabilities  $|\langle \phi_n | \psi_\alpha \rangle|^2$ , valid for *given* quantum states  $\psi_\alpha$ . In this way, the concept of a density matrix depends on the probability interpretation of the wave function – though *not* yet on any specific form in terms of ensembles<sup>2</sup> (see Sect. 4.6).

An ensemble interpretation of the density matrix according to  $\rho = \sum_\alpha |\psi_\alpha\rangle p_\alpha \langle \psi_\alpha|$ , used in (4.22), does not require the members  $\psi_\alpha$  of the ensemble of wave functions to be mutually orthogonal; they may even form an overcomplete set. The ensemble can therefore not be recovered from the density matrix. Von Neumann’s entropy (4.4) describes an ensemble entropy of the form  $S[\rho] = -k \sum p_\alpha \ln p_\alpha$  only for the specific ensemble consisting of the orthonormal eigenstates of  $\rho$ .

Just as for classical statistical mechanics, the conservation of entropy reflects dynamical determinism (now for wave functions) – provided the Hilbert state norm is conserved, too. This requires not only determinism, but also the *unitarity* of the Schrödinger equation (not just that of the von Neumann equation). The reason is that the formal density matrix cannot distinguish between the norm and probability  $p_\alpha$  of a wave function.

It should also be emphasized here that this formalism applies as well to wave functionals characterizing quantum field theory (that is, wave functions for a continuum of variables). ‘Backward running’ world lines in Feynman graphs are mere symbols for certain terms which appear in a relativistic perturbation expansion that is used for calculating the unitary propagation of wave functionals (general superpositions) with respect to an arbitrary but given time coordinate. These terms represent integrals over field modes (usually plane waves) – not over particle variables. Feynman’s approach has turned out to be useful even beyond  $S$ -matrix theory, which is restricted to describing interactions between asymptotically free objects.

The mapping of *general ensembles* of wave functions onto those which diagonalize the density matrix is an information-reducing idempotent operation

<sup>2</sup> If the elements of the probability interpretation are themselves wave functions (as in Born’s original formulation, mentioned in footnote 1, or as in collapse theories), the ensemble consisting of all possible outcomes of all conceivable measurements would be quite different from the initial ensemble (which may consist of one pure state, for example). Nonetheless, probabilities for all these outcomes are implicitly *postulated* by the phenomenological rules used in (4.22).

on these ensembles, similar to a Zwanzig projection. Nonetheless, one may rederive the von Neumann equation (4.1) from the ensemble interpretation under the further assumption that all wave functions defining the ensemble satisfy the same Schrödinger equation  $i\partial\psi_\alpha/\partial t = H\psi_\alpha$ . However, presuming the exact Hamiltonian to be ‘given’ is hardly consistent when regarding *states* as incompletely known. Even in classical physics, the precise Hamiltonian would depend on the (even less known) microscopic state of the environment (see Borel’s argument in Sect. 3.1.2).

Instead of representing an ensemble of wave functions, the density matrix may also describe the local (or ‘reduced’) perspective of *entangled* quantum systems, which are generically of the form

$$\psi(x, y) = \sum_{m,n} d_{mn} \phi_m(x) \Phi_n(y) . \quad (4.23)$$

For spatially separate subsystems, this entanglement defines *quantum nonlocality*. For example, it is responsible for the violation of Bell’s inequality (Bell 1964) or its stronger variants (Greenberger, Horne, Shimony and Zeilinger 1990), and it explains so-called *quantum teleportation* in a way which demonstrates that nothing has to be teleported: it must rather be *prepared* in advance as a component of an entangled state (see Zeh 2005c or Timpson 2005).

All measurements performed on a subsystem – corresponding to the states  $\phi(x)$ , say – of an entangled system can be characterized by the expectation values for all its subsystem observables  $A_\phi$ :

$$\langle A_\phi \rangle := \text{Trace}\{A_\phi \rho_{\text{total}}\} = \text{Trace}_\phi\{A_\phi \rho_\phi\} . \quad (4.24)$$

Here, the ‘reduced density matrix’  $\rho_\phi$  is defined as a partial trace,

$$\rho_\phi := \text{Trace}_\Phi\{\rho_{\text{total}}\} . \quad (4.25)$$

The total density matrix  $\rho_{\text{total}}$  may well be a pure state,  $\rho_{\text{total}} := |\psi\rangle\langle\psi|$ . The *new* density matrix  $\rho_\phi$  would then be explicitly given in terms of the expansion coefficients  $d_{mn}$  of the total state (4.23) as

$$(\rho_\phi)_{mm'} := \langle \phi_m | \rho_\phi | \phi_{m'} \rangle = \sum_n d_{mn} d_{m'n}^* , \quad (4.26)$$

rather than in terms of probabilities  $p_\alpha$ , which would instead lead to (4.8).

Both types of density matrices are Hermitean and positive by construction. They can therefore be diagonalized in the form  $\rho_\phi = \sum_n |\tilde{\phi}_n\rangle p_n \langle\tilde{\phi}_n|$ , with non-negative eigenvalues  $p_\alpha$ , in their eigenbasis  $\{\tilde{\phi}_n\}$ . This diagonal form defines a formal (or apparent) ensemble of orthonormal states. Although the LHS of (4.26) is thus identical with a density matrix describing an ensemble of (orthogonal or other) states, it is evident from the RHS that it does *not* represent one. Therefore, the ‘apparent ensemble’ or ‘improper mixture’ (d’Espagnat 1966) must not be used in an attempt to *explain* the probability

interpretation (4.24) on which it is based. The density matrix formalism is blind to the measurement problem (see below and Sect. 4.6).

For an entangled state such as (4.23), the eigenbases of the subsystem density matrices define the *Schmidt canonical form*,

$$\psi(x, y) = \sum_k \sqrt{p_k} \tilde{\phi}_k(x) \tilde{\Phi}_k(y) . \quad (4.27)$$

In contrast to the general representation (4.23) this is a single sum (Schmidt 1907, Schrödinger 1935). Phase factors for the coefficients  $\sqrt{p_k}$  have here been absorbed into the phase-ambiguity in the definition of the orthonormal states  $\tilde{\phi}_k$  or  $\tilde{\Phi}_k$ . For given subsystems, this representation (and hence its time dependence – see Kübler and Zeh 1973) is determined by the state  $\psi(t)$  of the total system – except for accidental degeneracy of the  $p_k$ 's.

The *neglect* of all correlations between two subsystems describes a specific loss of information, and so defines a new (nonlinear) Zwanzig projection,

$$\hat{P}_{\text{sep}}\rho := \rho_\phi \otimes \rho_\Phi . \quad (4.28)$$

A stronger Zwanzig projection of locality,  $\hat{P}_{\text{local}}\rho = \prod_k \rho_{\Delta V_k}$ , where the volume elements  $\Delta V_k$  form a complete set of local subsystems, would lead to a density matrix that factorizes, as in (3.38), in terms of these volume elements. It is again required in order to obtain the approximate concept of an *entropy density*  $s(\mathbf{r})$ . In contrast to this local picture, indistinguishable particles *cannot* be used to define subsystems that might give rise to a ‘substantial picture’. Therefore, the formal correlations between particles which describe symmetrization or antisymmetrization of the wave function does *not* represent any entanglement. These pseudo-correlations are merely an artifact from the use of classical particle concepts – see (4.21).

As a consequence of the nonlocality of quantum states, and in fundamental contrast to classical physics, the entropies  $S[\hat{P}_{\text{sep}}\rho]$  or  $S[\hat{P}_{\text{local}}\rho]$  of a completely defined (pure) quantum state are nontrivial: generically they do not vanish, since states of subsystems are *not defined* (rather than merely being unknown). *Apparent* ensembles, which are defined for them, may even be regarded as the *representative ensembles* used in statistical thermodynamics (see Chap. 3). However, one may now wonder (1) why microscopic systems are often found in pure states (such as eigenstates of their Hamiltonians  $H_0$ ), and (2) why the macroscopic world is successfully described by means of *given* classical concepts rather than in terms of their superpositions.

A local concept of relevance that, in contrast to  $\hat{P}_{\text{sep}}$ , preserves all ‘statistical’ correlations (those based on incomplete information), while removing all quantum correlations (entanglement), may be defined by using the Schmidt canonical representation in the form

$$\hat{P}_{\text{classical}}(|\psi\rangle\langle\psi|) := \sum_k p_k |\tilde{\phi}_k\rangle\langle\tilde{\phi}_k| \otimes |\tilde{\Phi}_k\rangle\langle\tilde{\Phi}_k| . \quad (4.29)$$