

Obituary



Heinrich-Wolfgang Leopoldt

22. 8. 1927 – 28. 7. 2011

Heinrich-Wolfgang Leopoldt, member of the editorial board of *Journal of Number Theory* from 1969 to 1987, has passed away on July 28, 2011 at the age of almost 84 years. His name will be remembered as the discoverer of the p -adic L -functions (jointly with T. Kubota) and as the author of the Leopoldt conjecture.

Leopoldt was born and raised in the small German town of Schwerin at the Baltic sea. Still a school boy he was drafted in war time to various military services. After the war he started an apprenticeship in view of the uncertain prospects of the future. However there was a former mathematics teacher with whom Leopoldt played music, and who introduced him also to astronomy and its mathematical fundamentals. He suggested to Leopoldt to finish his *Gymnasium* (high school) in order to be able to attend university. Leopoldt did so, and in the academic year 1947/48 he started his mathematical studies at the Humboldt University in Berlin.

One of the first lecture courses he attended was an introduction to number theory by Helmut Hasse. In his *Erinnerungen* Leopoldt reports that this lecture had left a deep impression upon him, in particular Hasse's remarks on the relationship between beauty and truth in mathematics, combined with

various instances of parallels between number theory and music. This lecture, Leopoldt recalls, led him to study number theory with highest priority. We can observe this dedication to number theory throughout his mathematical work.

In the year 1950 Hasse left Berlin for Hamburg and his student Leopoldt followed him there. 1954 he obtained his Ph.D. at Hamburg University. Thereafter he got a position as assistant professor in Erlangen until 1962, interrupted for two years at the Institute for Advanced Study in Princeton. 1959 he made his *Habilitation*, i.e., his qualifying examination as a university teacher. 1962-1964 he taught at Tübingen University, again interrupted by a visiting professorship at Johns Hopkins University in Baltimore. In the year 1964 he obtained an offer for a permanent position at Johns Hopkins and another one at Karlsruhe University. He accepted the latter.

Leopoldt's mathematical work is governed by the attempt "*to systematically open up the structure of abelian fields such that one can work in them freely in the same way as one can work in quadratic fields.*" This challenge had been formulated by Hasse in a monograph 1952, and his student Leopoldt set about to perform this task step by step. Here I cannot give an assessment of Leopoldt's complete work. But while I am writing this obituary I remember particularly three of his most brilliant achievements which had impressed me at the time – and I would like to share my recollections here.

First I remember his paper on the structure of the ring of integers in an abelian field K which was his *Habilitationsschrift* at Erlangen. If K is tamely ramified then Emmy Noether had shown that there exists a integral normal base, i.e., the ring of integers I of K is G -module isomorphic to the group ring $\mathbb{Q}[G]$, where G denotes the Galois group of K . However, if the ramification is not tame then there does not exist a normal integral base and it was not known how Noether's theorem should be modified in order to describe the G -structure of the integers. But Leopoldt's paper solves this problem, first in an abstract setting showing that I is G -module-isomorphic to a certain order of $\mathbb{Q}[G]$, and then interpreting this in a very explicit and concrete manner by constructing explicitly a normal basis with the help of the Gaussian sums $\tau(\chi)$ belonging to the characters χ of G .

Secondly I recall his brilliant paper, composed in Princeton, on the *Spiegelungssatz* in an absolute galois number field K . This concerns the structure of the class group C of K , more precisely: the structure of its p -part C_p where p is a fixed prime. Assume that the degree of K is prime to p and that the p -th roots of unity are in K . The action of the galois group G of K induces a direct decomposition of C_p into subgroups C_φ where φ ranges over the p -adically irreducible characters of G . Let ψ_0 denote the character belonging to the representation of G on the p -th roots of unity. Then there

is what Leopoldt calls the *reflection map* $\varphi \rightarrow \bar{\varphi}$ with

$$\bar{\varphi}(\sigma) = \psi_0(\sigma)\varphi(\sigma^{-1}) \quad (\text{for } \sigma \in G).$$

This construction of $\bar{\varphi}$ had arisen in various situations before but Leopoldt had discovered a relationship between the φ -part C_φ and the $\bar{\varphi}$ -part $C_{\bar{\varphi}}$. In some sense both are of about the same size. More precisely, if $e_\varphi, e_{\bar{\varphi}}$ denote the G -ranks of $C_\varphi, C_{\bar{\varphi}}$ respectively then

$$-\delta_{\bar{\varphi}} \leq e_{\bar{\varphi}} - e_\varphi \leq \delta_\varphi.$$

where the bounds $\delta_\varphi, \delta_{\bar{\varphi}}$ depend in a well defined manner on the p -adic galois structure of the group of units E of K . Hence the study of the galois structure of the units of K leads to information about the deviation of the rank of C_φ from that of its mirror image $C_{\bar{\varphi}}$.

Leopoldt's proof of this theorem uses class field theory; it is quite lucid and straightforward. It did not use any newly developed method and could have been proved much earlier. The essential new ingredient is the way of looking at the various objects from the structural point of view. Instead of dealing with *class numbers* Leopoldt deals with the *class groups* and their galois structure. This point of view had been advocated by Hasse in his monograph cited above, and Leopoldt now uses it competently to advance a big step towards the understanding of the class groups of number fields.

Nowadays this point of view, i.e., the investigation of class groups by means of galois action, has become standard in algebraic number theory, as it is manifested, e.g., in what is called the Iwasawa theory. At the time of publication the *Spiegelungssatz* won widespread interest among number theorists – not only because it offered a common background for classically known results about divisibility properties of class numbers, but also since it admitted to obtain much more information of this kind. Whereas classically (since Kummer) the properties of class numbers are connected with the Bernoulli numbers B^n , Leopoldt's new results referred to his generalized Bernoulli numbers B_χ^n belonging to Dirichlet characters χ . They are defined recursively in a similar way as are the ordinary Bernoulli numbers. Leopoldt showed that his B_χ^m , which are contained in the cyclotomic field $\mathbb{Q}(\chi)$, occur as the values of the Dirichlet L -functions $L(s, \chi)$ at the odd negative integers:

$$L(1 - m, \chi) = -\frac{B_\chi^m}{m}.$$

These relations lead to the third outstanding result of Leopoldt which I would like to recall, namely the discovery of the *p -adic L -functions*. They

belong to the standard tools of today's algebraic number theory but perhaps it is good to recall that they are due to Leopoldt. The story is as follows.

For any prime p his generalized Bernoulli numbers B_χ^m satisfy certain p -adic congruences, the so-called Kummer congruences. Leopoldt observed that those congruences can be interpreted such that the $\frac{B_\chi^m}{m}$ are in a sense p -adically continuous functions of m . More precisely: There is one and only one p -adically continuous function $L_p(s, \chi)$ defined on \mathbb{Z}_p such that

$$L_p(1 - m, \chi) = -\frac{B_\chi^m}{m}(1 - \chi(p)p^{m-1})$$

for the negative integers $1 - m$ with $m \equiv 0 \pmod{p-1}$.¹ These numbers $1 - m$ are dense in \mathbb{Z}_p . But it turns out that $L_p(s, \chi)$ is holomorphic in a region which is larger than \mathbb{Z}_p , at least if $\chi \neq 1$ whereas for $\chi = 1$ there is one pole for $s = 1$.

Based on these p -adic L -functions Leopoldt considered for any abelian number field K the corresponding p -adic zeta function $\zeta_{K,p}(s)$ as the product of the $L_p(s, \chi)$ for the characters χ of K . He arrives at p -adic class number formulas which are the analogue of the ordinary class number formulas for abelian fields K , and they look quite similar but the terms have to be interpreted in the p -adic sense.

However there was one obstacle to overcome, namely the proof of non-vanishing of the so-called p -adic regulator of an abelian field which appears in those formulas. This regulator $R_{K,p}$ is obtained if one replaces the ordinary logarithms in the classical regulator by the p -adic logarithms. The non-vanishing of $R_{K,p}$ means that the p -adic rank of the group of units of K equals its ordinary rank, i.e., $r_1 + r_2 - 1$ where r_1, r_2 are the numbers of real or complex infinite primes respectively.

Leopoldt had tried hard to prove this conjecture. He delayed the publication of the second part of his L -series paper since he still hoped to be able to include a proof of it. Nevertheless he presented an exposition of his theory when he lectured at Johns Hopkins in the year 1964. In this form the theory somehow made it to Princeton where Iwasawa included it in his *Lectures on p -adic L -functions*. This made Leopoldt's theory widely known and started an extended research in consequence of which p -adic L -series became indispensable tools of number theorists. Brumer succeeded 1967 to prove Leopoldt's conjecture for arbitrary abelian fields.²

A characteristic feature of Leopoldt's work is that he aims at concrete results given by explicit and effective formulas. He competently uses and

¹For simplicity we assume here that χ is an even character and that $p > 2$.

²Recently Mihailescu has announced a proof of Leopoldt's conjecture for all number fields.

investigates abstract structures, but such considerations serve him as motivation only, as guideline towards the goal of explicit algorithms. To a large degree his results were obtained by extended numerical computations.

This attitude led him quite early to develop computer programs for the use in algebraic number theory. His team in Karlsruhe was one of the first in Germany which systematically developed the necessary algorithms for this project, also in cooperation with Hans Zassenhaus. A number of Leopoldt's students are now working in scientific computing and so follow up his ideas.

Leopoldt's personality can be characterized as quiet, unassuming, always willing to hold back his own in favor of supporting the cause. His firm and objective counsel in scientific matters, always to the point, was valued by all who had to deal with him. His lectures stood out by their clarity and intensity. He was known as a master in exposition. The list of his publications is not large but his work belongs to the pearls of mathematical research in the last century.

Leopoldt was married and has five children. After his retirement from the University of Karlsruhe he had moved to a small village in Northern Germany where he dedicated himself to his beloved piano music.

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